Maass forms and the mock theta function $f(q)$

Alexander Dunn

Joint work with Scott Ahlgren
University of Illinois at Urbana–Champaign

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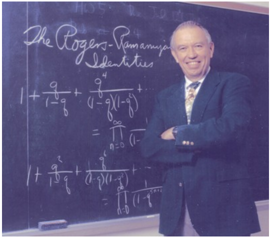


Ranks

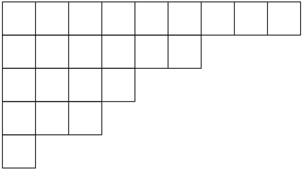
- Dyson (1944) attempted to combinatorially explain Ramanujan's congruences

$$p(5n+4) \equiv 0 \pmod{5}, \quad p(7n+5) \equiv 0 \pmod{7}$$

$$p(11n+6) \equiv 0 \pmod{11},$$

by introducing the **rank** of a partition.

$$\text{rank of a partition} = \text{largest part} - \text{number of parts}.$$



- The rank of this partition of 23 is $9 - 5 = 4$.

Mock theta function $f(q)$

- Define:

$$\mathcal{N}_e(n) = \text{number of partitions of } n \text{ of even rank}$$
$$\mathcal{N}_o(n) = \text{number of partitions of } n \text{ of odd rank}$$

- The generating function for the difference

$$\alpha(n) := N_e(n) - N_o(n),$$

is Ramanujan's famous third order mock theta function:

$$f(q) := 1 + \sum_{n=1}^{\infty} \alpha(n) q^n$$
$$= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2}.$$

Andrews' 1964 PhD thesis

Andrews proved the remarkable formula (valid for any $\epsilon > 0$):

$$\alpha(n) = \frac{\pi}{(24n-1)^{\frac{1}{4}}} \sum_{c=1}^{\lfloor\sqrt{n}\rfloor} \frac{(-1)^{\lfloor \frac{c+1}{2} \rfloor} A_{2c}\left(n - \frac{c(1+(-1)^c)}{4}\right)}{c} \times I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12c}\right) + O_\epsilon(n^\epsilon).$$

- $I_{\frac{1}{2}}$ is the I–Bessel function of order $1/2$
- $A_c(n)$ is the generalised Kloosterman sum

$$A_c(n) := \sum_{\substack{d \pmod{c} \\ (d,c)=1}} e^{\pi i s(d,c)} e\left(-\frac{dn}{c}\right), \qquad c, n \in \mathbb{N}.$$

- $s(d,c)$ is the usual Dedekind sum and $e(x) := \exp(2\pi i x)$.

Andrews' first conjecture...

Conjecture 1 (Andrews,1966)

$$\alpha(n) = \frac{\pi}{(24n-1)^{\frac{1}{4}}} \sum_{c=1}^{\infty} \frac{(-1)^{\lfloor \frac{c+1}{2} \rfloor} A_{2c}\left(n - \frac{c(1+(-1)^c)}{4}\right)}{c} \times I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12c}\right).$$

- Conjecture 1 was resolved by Bringmann and Ono in 2006.
- Uses Zwegers work on vector valued modular forms.

Truncation and remainder

- We return to the classical problem obtaining a power savings error when one truncates the exact for $\alpha(n)$.
- Define the error term $R(n, N)$ by

$$\alpha(n) = \frac{\pi}{(24n-1)^{\frac{1}{4}}} \sum_{c \leq N} \quad \cdots \quad + \quad R(n, N).$$

- Result of Andrews gives

$$R(n, \sqrt{n}) \ll_{\epsilon} n^{\epsilon}.$$

- We break this ε-barrier (prove a negative exponent).

Power saving error for coefficients of $f(q)$

Theorem 2 (Ahlgren, D., 2018)

Suppose that $24n - 1$ is positive and squarefree. Then for all $\varepsilon > 0$ and $\gamma > 0$ we have

$$R(n, \gamma\sqrt{n}) \ll_{\gamma,\epsilon} n^{-\frac{1}{147}+\varepsilon}.$$

Some perspective with famous conjectures

- Ramanujan–Lindelöf conjecture for the coeffs of weight $1/2$ Maass cusp forms:

$$R(n, \gamma\sqrt{n}) \ll_{\gamma,\epsilon} n^{-\frac{1}{16}+\epsilon}.$$

- Linnik–Selberg conjecture for the sums of Kloosterman sums:

$$R(n, \gamma\sqrt{n}) \ll_{\gamma,\epsilon} n^{-\frac{1}{4}+\epsilon}.$$

Power saving error for the partition function

- $\alpha(n)$ is reminescent of Rademachers' series:

$$p(n) = \frac{2\pi}{(24n-1)^{\frac{3}{4}}} \sum_{c=1}^{\infty} \frac{A_c(n)}{c} I_{\frac{3}{2}} \left(\frac{\pi\sqrt{24n-1}}{6c} \right)$$

- Define $S(n, N)$ by

$$p(n) := \frac{2\pi}{(24n-1)^{\frac{3}{4}}} \sum_{c \leq N} \quad \cdots \quad + \quad S(n, N).$$

- We improve a recent result of Andersen and Ahlgren.

Theorem 3 (Ahlgren, D., 2018)

Suppose that $24n-23$ is positive and squarefree. Then for all $\epsilon > 0$ and $\gamma > 0$ we have

$$S(n, \gamma\sqrt{n}) \ll_{\gamma,\varepsilon} n^{-\frac{1}{2}-\frac{1}{147}+\varepsilon}.$$

Kloosterman sum with a multiplier on $\Gamma_0(2)$

- $\frac{1}{2}$-integral weight eta multiplier is defined by:

$$\eta(\gamma\tau) = \nu_\eta(\gamma)\sqrt{c\tau + d}\,\eta(\tau), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

- For $\gamma \in \Gamma_0(2)$, define

$$\psi(\gamma) = \begin{cases} i^{c/2}\left(\frac{-1}{d}\right)\overline{\nu_\eta}(\gamma) & \text{if} \quad c \equiv 0 \pmod 4, \\ i^{c/2}\overline{\nu_\eta}(\gamma) & \text{if} \quad c \equiv 2 \pmod 4. \end{cases}$$

- Generalised Kloosterman sum attached to ψ:

$$S(m, n, c, \psi) := \sum_{\substack{0 \le a, d < c \\ \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)}} \overline{\psi}(\gamma) e\left(\frac{m_\psi a + n_\psi d}{c}\right).$$

- Here, $n_\psi := n - \frac{1}{24}$.

Summands of $\alpha(n)$

- For $c > 0$ we have

$$(-1)^{\lfloor \frac{c+1}{2} \rfloor} A_{2c}\left(n - \frac{c(1+(-1)^c)}{4}\right) = e(\frac{1}{8})\overline{S(0,n,2c,\psi)}.$$

- Thus proving power savings bound is really about showing cancellation amongst sums of generalised Kloosterman sums.
- Appeal to the spectral theory of $\frac{1}{2}$-integral weight automorphic forms.

Spectral theory of automorphic forms

- For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ and $\tau = x + iy \in \mathbb{H}$, define

$$j(\gamma, \tau) := \frac{c\tau + d}{|c\tau + d|} = e^{i \arg(c\tau+d)}.$$

- Weight $\frac{1}{2}$ Laplacian:

$$\Delta_{\frac{1}{2}} := y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - \frac{1}{2} iy \frac{\partial}{\partial x}$$

- Denote by $\mathcal{L}_{\frac{1}{2}}(2, \psi)$ the space of L^2–functions on $\Gamma_0(2)\backslash\mathbb{H}$ which satisfy

$$f(\gamma\tau) = j(\gamma, \tau)^{\frac{1}{2}} \psi(\gamma) f(\tau) \quad \text{for all} \quad \gamma \in \Gamma_0(2).$$

Spec. theory of aut. forms cont...

- Consider
$$\Delta_{\frac{1}{2}} u_j + \lambda_j u_j = 0.$$
- The spectrum of $\Delta_{\frac{1}{2}}$ on $\mathcal{L}_{\frac{1}{2}}(2, \psi)$ is discrete, countable and has only limit point ∞:
$$\lambda_j = \frac{1}{4} + r_j^2 \quad r_j \in i\big(0, \frac{1}{4}\big] \cup [0, \infty),$$
$$u_j(\tau) = c_0(y) + \sum_{n_\psi \neq 0} \rho_j(n) W_{\frac{\mathrm{sgn}(n_\psi)}{4}, ir_j}(4\pi |n_\psi| y) e(n_\psi x),$$
- Always have $c_0(y) = 0$, so we can take an orthonormal basis consisting of Maass cusp forms (lift of Sarnak).

Kuznetsov formula

- Let $\phi : [0, \infty) \to \mathbb{C}$ be an appropriate test function and $\check{\phi}$ be a certain K-Bessel transform.
- Kuznetsov trace formula:

$$\sum_{\substack{c>0 \\ c\equiv 0 \pmod 2}} \frac{S(0, n, c, \psi)}{c} \phi\left(\frac{2\pi n_\psi}{\sqrt{6}c}\right) = \frac{4\sqrt{i}}{\sqrt{6}} \sqrt{n_\psi} \sum_{r_j} \frac{\overline{\rho_j(0)}\rho_j(n)}{\mathrm{ch}\pi r_j} \check{\phi}(r_j).$$

- Sums of Kloosterman sums $\rightsquigarrow$ Bounding Fourier coefficients

Strategy

- Delicate double dyadic decomposition of the right hand side in both the modulus c and spectral parameter r_j.
- Bounds for both J and K Bessel integral transforms
- Three different bounds for the Fourier coefficients $\rho_j(n)$ of Maass cusp forms.
- Small and large r_j are handled using estimates of Duke–Andersen and Ahlgren–Andersen.
- Intermediate sized r_j are handled using new estimates of Ahlgren and D.

Spectral estimates and the theta multiplier

- Let ν_θ be the multiplier of weight $1/2$ on $\Gamma_0(4)$ associated to the theta function.
- For each r, the map $\tau \mapsto 24\tau$ gives an injection

$$\mathcal{L}_{\frac{1}{2}}(2, \psi, r) \longrightarrow \mathcal{L}_{\frac{1}{2}}\left(144, \left(\frac{12}{\bullet}\right)\nu_\theta, r\right).$$

- Sufficient to have bounds for the coefficients of Maass cusp forms attached to the quadratically twisted theta multiplier.
- Our bounds are stronger in the spectral aspect than those of Duke and Baruch–Mao (with the cost of a slight loss in the n-aspect).

Set-up

- Let N be a positive integer with $4 \mid N$ and let D be an even discriminant.
- Suppose that $f(\tau)$ is a smooth function on $\mathbb{H}$ that satisfies

$$f(\gamma\tau) = j(\gamma,\tau)^{\frac{1}{2}} \left(\frac{|D|}{d}\right) \nu_\theta f(\tau) \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$$

  and

$$\int_{\Gamma_0(N)\backslash\mathbb{H}} |f(\tau)|^2 \, \frac{dxdy}{y^2} = 1.$$

- Suppose

$$\Delta_{\frac{1}{2}} f + \lambda f = 0, \qquad \lambda = \frac{1}{4} + r^2$$

- Fourier expansion:

$$f(\tau) = c_0(y) + \sum_{n\neq 0} \rho(n) W_{\frac{1}{4}\,\mathrm{sgn}(n), ir}(4\pi|n|y) e(nx).$$

Our estimate

Theorem 4 (Ahlgren, D., 2018)

With notation as above, suppose that $f(\tau)$ satisfies the conditions above. Then for square free $n \neq 0$ we have

$$\rho(n) \ll_{N,\epsilon} \lambda^{\frac{3}{4}-\frac{\operatorname{sgn} n}{8}} ch\frac{\pi r}{2}|n|^{-\frac{163}{588}+\epsilon}.$$

- Delicate argument involving a Kuznetsov formula of Duke, Friedlander and Iwaniec.
- Problematic terms arise from an infinite series involving coefficients of holomorphic cusp forms.

Coefficients of $f(q)$ and quadratic forms (set-up)

- We obtain asymptotic formula for $\alpha(n)$ as a sum over a set of quadratic points in the upper half-plane $\mathbb{H}$.
- Similar in spirit to a result of Masri who makes use of a formula of Alfes for $\alpha(n)$.
- Suppose that $D > 0$ and define

$$\mathcal{Q}_{-D,12} := \left\{ ax^2 + bxy + cy^2 : b^2 - 4ac = -D,\ 12 \mid a,\ a > 0 \right\}.$$

- $\Gamma_0(12)$ acts on this set from the left and preserves $b \pmod{12}$.
- If $Q = [12a, b, c] \in \mathcal{Q}_{-D,12}$, we define $\chi_{-12}(Q) = \left(\frac{-12}{b}\right)$.
- Let τ_Q denote the root of $Q(\tau, 1)$ in $\mathbb{H}$.

Our theorem

Theorem 5 (Ahlgren, D., 2018)

Suppose that $24n-1$ *is positive and squarefree. Then for all* $\varepsilon > 0$ *and* $\gamma > 0$ *we have*

$$\alpha(n) = \frac{i}{\sqrt{24n-1}} \sum_{\substack{Q \in \Gamma_\infty \backslash \mathcal{Q}_{1-24n,12} \\ \mathrm{Im}\tau_Q > \gamma}} \chi_{-12}(Q) \left(e(\tau_Q) - e(\overline{\tau}_Q) \right) + O_{\gamma,\varepsilon}\left(n^{-\frac{1}{147}+\varepsilon} \right).$$

Andrews second conjecture...

Conjecture 6 (Andrews, 1966)

The series representation for $\alpha(n)$ does not converge absolutely for any value n.

Theorem 7 (Ahlgren, D., 2018)

Andrews' second conjecture is true.

Resolution of Andrews' second conjecture

- Use Fourier analysis and Gauss sums (involved computation) to write Kloosterman sums as Weyl–type sums:

$$\overline{S(0, n, 2c, \psi)}. = \sqrt{\frac{c}{24}} e\left(\frac{1}{8}\right) F_{2c}(n)$$

where

$$F_{2c}(n) := \sum_{\substack{x \pmod{48c} \\ x^2 \equiv 1-24n \pmod{48c}}} \left(\frac{-12}{x}\right) e\left(\frac{x}{12c}\right)$$

- Let $p \geq 5$ be prime such that

$$\left(\frac{1-24n}{p}\right) = 1.$$

- For such a p let m_p satisfy

$$48^2 m_p^2 \equiv 1 - 24n \pmod{p}.$$

Andrews' second conjecture cont.

- $F_{2p}(n)$ is large "often".
- We have evaluation

$$F_{2p}(n) = 2\sqrt{24}\, i(-1)^n \left(\frac{-12}{p}\right) e\left(\frac{p^2-1}{48}\right) \cos\left(\frac{4\pi m_p}{p}\right).$$

- Equidistibution theorem of Duke, Friedlander and Iwaniec of quadratic residues to prime moduli tells you that the cosine is large for a positive proportion of primes!