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Nahm's conjecture in the Cartan case

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Abstract

When is a q-hypergeometric series a modular form? The connection between these two objects dates back to 1913 in Ramanujan's first letter to Hardy. In general, this is a deep question and out of reach at present. In 1994, Nahm conjectured a criterion for this phenomenon that has become a guiding principle in this area of research. The conjecture connects the modularity of a family of Eulerian series to torsion elements in Bloch groups of number fields determined by the Eulerian series. In 2011, Vlasenko and Zwegers exhibited counter-examples to this conjecture. Despite this, a theorem of Lee in 2013 provides strong evidence the conjecture is true in a special case related to models in conformal field theory. These models are parameterised by a pair (X, X'), where X and X' are Dynkin diagrams of ADET type. When $\gcd\big((n-1)!, k\big) = 1$, we prove Nahm's conjecture holds in the case $(X, X') = (\mathrm{A}_{n-1}, \mathrm{A}_{k-1})$. We make use of string functions coming from the representation theory of affine Kac–Moody algebras. This complements previous results pertaining to $(X, X') = (A_{2n}, T_{k-1})$ due to Feigin–Stoyanovsky ($n = 1$) and Stoyanovsky, and $(X, X') = (A_{2n-1}, T_1)$ due to Warnaar–Zudulin. We also conduct computational investigations in classical Andrews–Gordon case i.e. when $(X, X') = (\mathrm{A}_1, \mathrm{T}_{k-1})$. In particular, Keegan and Nahm ask whether a special family of modular B-vectors are the only ones giving rise to a modular Eulerian series. We confirm this for $k = 3$ and $k = 4$.

Chapter 2 will introduce the basic objects needed to state Nahm's conjecture. This will include details on modular forms, dilogarithms and the Bloch group.

In Chapter 3 we will provide a statement of Nahm's conjecture, its asymptotic motivation and counter-examples due to Vlasenko and Zwegers.

In Chapter 4 we adopt a less number theoretic point of view and first discuss affine Kac–Moody algebras and some of their representation theory. From this theory we have access to the string functions, their modularity properties and the Virasoro algebra. The end of this chapter contains the original work of the author.

Declaration by author

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Publications during candidature

None.

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Statement of parts of the thesis submitted to qualify for the award of another degree

None.

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Keywords

Modular form/function, dilogarithm, Bloch group, torsion, hypergeometric series, Kac–Moody algebra, string function, minimal model

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Contents

Chapter 1

Introduction

An important problem in number theory is to understand the intersection between basic or q-hypergeometric series and modular functions. In his first letter to Hardy [Har59, p. 9], Ramanujan wrote down the striking identity

$$\frac{1}{1+\cfrac{e^{-2\pi}}{1+\cfrac{e^{-4\pi}}{\ddots}}} = \left(\sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}\right) e^{2\pi/5}. \tag{1.0.1}$$

One can witness a beautiful interaction between q-series and modular functions by looking at the origins of this identity. In particular, (1.0.1) is obtained using the Rogers–Ramanujan identities

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \prod_{n\equiv \pm 1 \pmod 5}^{\infty} \frac{1}{1-q^n} \tag{1.0.2a}$$

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n} = \prod_{n\equiv \pm 2 \pmod 5}^{\infty} \frac{1}{1-q^n}, \tag{1.0.2b}$$

$|q| < 1$. The left hand sides of (1.0.2) are q-hypergeometric series. These are series of the form $\sum_{n=0}^{\infty} A_n(q)$ where $A_0(q) = 1$, $A_n(q) = R(q, q^n)A_{n-1}(q)$ for all $n \geq 1$ where $R(x, y)$ is a rational function with $\lim_{y\to 0} R(x, y) = 0$. These types of series have long been studied in connection with the theory of integer partitions and combinatorics [Mac60, And86, And98, GR04]. Writing $q = e^{2\pi i\tau}$ with $\tau \in \mathbb{H}$, the right hand sides of (1.0.2) are modular functions up to a rational power of q. These are functions characterized by the invariance under $\tau \mapsto (a\tau + b)/(c\tau + d)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ belonging to a finite-index subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Modular functions and their generalisations have been studied extensively in number theory and complex analysis [Ono04].

The main point is that the Rogers–Ramanujan identities appear in an unstructured list of q-hypergeometric series that are modular [Sla52]. The problem of describing the phenomenon of when a q-hypergeometric series is modular is both deep, and out of reach at present.

Nahm's conjecture provides a guiding principle for a special case of this problem. He conjectured a criterion for the modularity of the r-fold Eulerian series

$$F_{A,B,C}(q) := \sum_{n\in(\mathbb{Z}_{\geq 0})^r} \frac{q^{\frac{1}{2}n^TAn+Bn+C}}{(q)_{n_1}\cdots(q)_{n_r}}, \qquad q = \mathrm{e}^{2\pi i\tau}, \quad \tau\in\mathbb{H}, \tag{1.0.3}$$

where $A \in M_r(\mathbb{Q})$ is an $r \times r$ positive-definite symmetric matrix, $B \in \mathbb{Q}^r$ and $C \in \mathbb{Q}$. Here $(x; q)_n := \prod_{k=0}^{n-1}(1 - xq^k)$ and $(q)_n := (q; q)_n$. For the exact statement of the conjecture see Section 3.1. Notice $G(q)$ and $H(q)$ are of the form (1.0.3) with $A = (2)$ and $B = (0),(1)$ respectively. We say (A, B, C) is a modular triple if $F_{A,B,C}(q)$ is modular function for some integer weight.

Nahm's conjecture draws upon concepts from number theory, algebraic K-theory, conformal field theory, cluster algebras and the representation theory of Kac–Moody algebras. Here we attempt to illustrate this claim with the toy q-hypergeometric series

$$F_{a,b,0}(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}an^2+bn}}{(q)_n}. \tag{1.0.4}$$

The asymptotic behaviour of modular forms as q approaches a root a unity is well understood. In particular, if $F(q)$ is a modular form of weight k with respect to some congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ and $q = \mathrm{e}^{-\varepsilon}$, then

$$F(q) \sim \mathrm{e}^{-k+r\pi^2/\varepsilon} \qquad \text{as} \qquad \varepsilon \searrow 0, \tag{1.0.5}$$

for some $r \in \mathbb{Q}$. In order for $F_{a,b,0}(q)$ to be a modular function for some weight, its asymptotics as $q \to 1^-$ must necessarily match those of modular forms. For $(a, b) \in \mathbb{Q}_+ \times \mathbb{Q}$ we have

$$\sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}an^2+bn}}{(q)_n} \sim \mathrm{e}^{L(x)/\varepsilon} \qquad \text{as} \qquad \varepsilon \searrow 0, \tag{1.0.6}$$

where x is the unique positive solution to the equation $x = (1 - x)^a$ and $L(x)$ is the Rogers dilogarithm defined in Section 2.3. Comparing (1.0.6) and (1.0.5), $L(x)$ is constrained to be a rational multiple of $L(1) = \pi^2/6$ and the weight of $F_{a,b,0}(q)$ must be $k = 0$. This suggests a connection to the Bloch group of the number field $F = \mathbb{Q}(x)$, denoted $\mathcal{B}(F)$. The linear combinations of arguments in (2.4.1) are used to describe the main functional equations for $L(x)$ in (2.3.3), where the right hand side has values that are rational multiples of $\pi^2/6$. This provides evidence that $[x] \in \mathcal{B}(F)$ is a torsion element. Nahm's conjecture largely generalises this idea, linking the modularity of $F_{A,B,C}(q)$ to torsion elements in Bloch groups determined by A.

Aside from asymptotics and algebraic K-theory, Nahm largely motived his conjecture using integrable perturbations of conformal field theories. In fact, all of the modular triples of rank one (q-hypergeometric series of the form (1.0.6) up to a rational power of q) classified in Theorem 3.2.1 correspond to characters coming from representations of the Virasoro algebra. For example in the Rogers–Ramanujan case we have the modular triples $(2, 0, -1/60)$ and $(2, 1, 11/60)$. In these cases, the unique positive solution to $x = (1-x)^2$ is $(3-\sqrt{5})/2$ and

$$L\left(\frac{3-\sqrt{5}}{2}\right) = \frac{2}{5}L(1).$$

The number $2/5$ corresponds to the effective central charge of the corresponding minimal model coming from conformal field theory. These aspects will be explained more in Section 4.7.

Certain integrable systems in mathematical physics can be described by a pair of Dynkin diagrams (X, Y) where X and Y are of ADE or T type (i.e. tadpole). Letting C_X denote the Cartan matrix of X, such models are described by the equations $Q = (1-Q)^A$ where $A = C_X \otimes C_Y^{-1}$. The effective central charge of each model is given by

$$c_{\text{eff}}(X, Y) = \frac{r_X r_Y h_X}{h_X + h_Y},$$

where r_X and h_X denote the rank and Coxeter number of the semi-simple Lie algebra with Dynkin diagram X [KN11]. It has been conjectured that choosing any A of the above form will yield at least one B and C such that (A, B, C) is a modular triple [Lee13]. A theorem of Lee [Lee13, Theorem 1.3] shows that when A is chosen of the above form, the first condition of Nahm's Conjecture 3.1.5 is satisfied. This provides strong evidence that there is at least one modular (A, B, C) in each case. In this spirit, it is interesting to ask whether Nahm's conjecture is true when we restrict to matrices of the form $A = C_X \otimes C_{X'}^{-1}$? We will consider part of this question in Chapter 4 in connection with the representation theory of Kac–Moody algebras. In particular, we show Nahm's conjecture holds when $(X, X') = (\mathrm{A}_{n-1}, \mathrm{A}_{k-1})$ subject to the condition $\gcd((n-1)!, k) = 1$, see Proposition 4.0.1. For this choice of X and X' there is a rich structure to the modular B-vectors related to the columns of the matrix $-C_{\mathrm{A}_{k-1}}^{-1}$. This structure can be witnessed in Georgiev's formula for the string functions of a specific family of highest weight A_{n-1}-modules of level k [Geo94]. We make use of Georgiev's formula in the proof of Proposition 4.0.1 in Section 4.10.

We also investigate the modular B-vectors of the classical Andrews–Gordon series i.e. when $(X, X') = (\mathrm{A}_1, \mathrm{T}_{k-1})$. Modularity for a simple family of B-vectors may be established by comparing the appropriate Nahm-type series to characters arising from the $M(2, 2k+1)$-minimal model. Keegan and Nahm in [KN11] ask whether these are the only modular B-vectors? For small values of k, they carry out a computational search in [KN11] for all modular B-vectors within the range $-8 \leq b_i \leq 8$ and only vectors in the simple family were obtained. We are able to confirm that this phenomenon happens with no restrictions on the b_i when $k = 3$ and $k = 4$, see Section 4.9.

Chapter 2

Preliminaries and Background

We first cover the basic ideas from number theory and related areas needed to motivate and state Nahm's conjecture in Section 3.1.

2.1 Modular functions and forms

We follow the exposition given in [Ste07]. As usual, denote the special linear group by

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1 \right\},$$

and the complex upper half plane

$$\mathbb{H} := \{\tau \in \mathbb{C} : \mathrm{Im}\tau > 0\}.$$

We will refer to $\mathrm{PSL}_2(\mathbb{Z}) := \mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$ as the modular group. This group acts on $\mathbb{H}$ by

$$\tau \mapsto \gamma(\tau) = \frac{a\tau + b}{c\tau + d}, \quad \gamma \in \mathrm{PSL}_2(\mathbb{Z}), \quad \tau \in \mathbb{H}.$$

The modular group has presentation $\langle S, T | S^2 = (ST)^3 = 1 \rangle$ where

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

These correspond to the mappings of $\mathbb{H}$

$$\tau \mapsto -\frac{1}{\tau} \quad \text{and} \quad \tau \mapsto \tau + 1,$$

respectively.

Figure 2.1 shows the fundamental domain $\mathcal{F}$ of the action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathbb{H}$. It is given by

$$\mathcal{F} = \left\{ \tau \in \mathbb{H} : |\tau| \geq 1 \text{ and } |\mathrm{Re}\tau| \leq \frac{1}{2} \right\}.$$

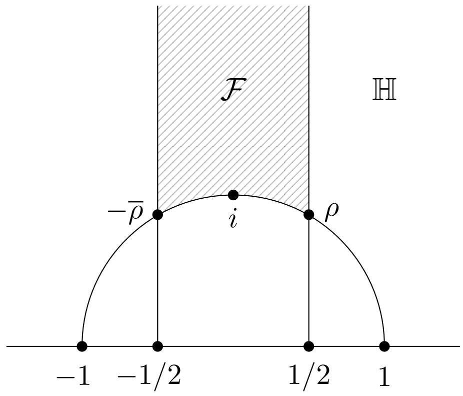

Figure 2.1: The fundamental domain $\mathcal{F}$.

Definition 2.1.1. Let $k \in \mathbb{Z}_{>0}$. A function $f : \mathbb{H} \to \mathbb{C}$ is weakly modular of weight k if it is meromorphic on $\mathbb{H}$ and satisfies

$$f(\gamma(\tau)) = (c\tau + d)^k f(\tau) \quad \text{for all} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}).$$

Since $\mathrm{PSL}_2(\mathbb{Z})$ is generated by S and T, to show that a meromorphic function f is weakly modular of weight k it is sufficient to establish

$$f(\tau + 1) = f(\tau) \quad \text{and} \quad f\left(-\frac{1}{\tau}\right) = \tau^k f(\tau).$$

Suppose f is weakly modular of weight k. If a Fourier expansion of f exists, it takes the form

$$f(\tau) = \sum_{n=m}^{\infty} a_n \mathrm{e}^{2\pi i n\tau}, \quad \tau \in \mathbb{H}.$$

Define the holomorphic function $q(\tau) := q = \mathrm{e}^{2\pi i\tau}$ on $\mathbb{C}$ and let D be the punctured open unit disk with origin removed. Observe that q restricts to the map $q : \mathbb{H} \to D$. Since $f(\tau+1) = f(\tau)$, there is a function F such that $F(q(\tau)) = f(\tau)$. Suppose that F is well-behaved at 0, in the sense that it permits a Laurent expansion about 0,

$$F(q) = \sum_{n=m}^{\infty} a_n q^n.$$

If this is the case, we say f is meromorphic at ∞. If $m \geq 0$, we say f is holomorphic at infinity.

Definition 2.1.2. A weakly modular function of weight k is called modular function of weight k if it is meromorphic at infinity.

Remark 2.1.3. We say a function is modular if it is a modular function of weight 0.

Definition 2.1.4. A modular function of weight k is a called a modular form of weight k if it is holomorphic everywhere including infinity.

When f is a modular form, there exist a_n such that

$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n \quad \text{for all} \quad \tau \in \mathbb{H}.$$

We have defined modular forms with respect to the full modular group. We now wish to define modular forms with respect to special subgroups of $\mathrm{PSL}_2(\mathbb{Z})$, called congruence subgroups. A congruence subgroup Γ of $\mathrm{PSL}_2(\mathbb{Z})$ is any subgroup that contains

$$\Gamma(N) = \ker\big(\mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/N\mathbb{Z})\big),$$

for some positive integer N. The minimum N is called the level of Γ.

We define a weight-k right action of $\mathrm{GL}_2(\mathbb{Q})$ on the set of all functions $f : \mathbb{H} \to \mathbb{C}$ by

$$f^{[\gamma]_k}(\tau) = \det(\gamma)^{k-1}(c\tau + d)^{-k} f(\gamma(\tau)),$$

where

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}).$$

This induces a right action of $\mathrm{GL}_2(\mathbb{Z})$ on the same set of functions. This restricted action has the the property

$$f^{[\gamma_1\gamma_2]_k} = \left(f^{[\gamma_1]_k}\right)^{[\gamma_2]_k},$$

for all $\gamma_1, \gamma_2 \in \mathrm{GL}_2(\mathbb{Z})$.

Definition 2.1.5. A weakly modular function of weight k with respect to a congruence subgroup Γ is a meromorphic function $f : \mathbb{H} \to \mathbb{C}$ such that

$$f^{[\gamma]_k} = f$$

for all $\gamma \in \Gamma$.

Lemma 2.1.6. Let $f : \mathbb{H} \to \mathbb{C}$ be a weakly modular function of weight k for a congruence subgroup Γ and $\delta \in \mathrm{PSL}_2(\mathbb{Z})$. Then $f^{[\delta]_k}$ is a weakly modular function for $\delta^{-1}\Gamma\delta$.

We are now ready to define a modular form with respect to an arbitrary congruence subgroup Γ.

Definition 2.1.7. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of weight k with respect to a congruence subgroup Γ if

(a) f is weakly modular with respect Γ

(b) $f^{[\gamma]_k}$ is meromorphic at ∞ for all $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$ i.e. it has a Fourier expansion of the form

$$f^{[\gamma]_k}(\tau) = \sum_{n=m}^{\infty} a_n q^{n/H}$$

for some integers $(m, H) \in \mathbb{Z} \times \mathbb{Z}_{>0}$.

Remark 2.1.8. We denote by $M_k(\Gamma)$ the $\mathbb{C}$-vector space of modular forms with respect to Γ .

Example 2.1.9. A famous modular function that will frequently appear in the rest of this thesis is the Dedekind η-function given by

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) = q^{1/24}(q)_\infty.$$

It has the modular transformation properties

$$\eta(\tau+1) = \mathrm{e}^{\pi i/12}\eta(\tau)$$
$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau),$$

with respect to the full modular group. Note $\eta(\tau)^{24}$ is thus a modular form of weight 12.

2.2 Asymptotics of modular forms

The number theoretic motivation for Nahm's conjecture is a comparison of asymptotics between $F_{A,B,C}(q)$ and a modular form. Modular forms have a special type of asymptotic expansion used in Chapter 3, and this expansion is given in the following lemma.

Lemma 2.2.1. Let $F(q) \neq 0$ be a modular form of weight k with respect to a congruence subgroup Γ. Then there exist $a \in \pi^2\mathbb{Q}$ and $0 \neq b \in \mathbb{C}$ such that

$$\mathrm{e}^{a/\varepsilon}F(\mathrm{e}^{-\varepsilon}) \sim b\varepsilon^{-k} + o(\varepsilon^N) \quad \text{for all} \quad N \geq 0. \tag{2.2.1}$$

Proof Since $F(q) \in M_k(\Gamma)$, the function $\tau^{-k}F\left(\mathrm{e}^{-2\pi i/\tau}\right) \in M_k(S\Gamma S)$ by Lemma 2.1.6. This function has q-expansion

$$\frac{1}{\tau^k}F\left(\mathrm{e}^{-2\pi i/\tau}\right) = \sum_{n=0}^{\infty} a_n q^{\alpha_n} \tag{2.2.2}$$

where $\alpha_n \in \mathbb{Q}$ and $\alpha_n < \alpha_{n+1}$. Substituting $\tau = 2\pi i/\varepsilon$ we obtain

$$F\left(\mathrm{e}^{-\varepsilon}\right) = \left(\frac{2\pi i}{\varepsilon}\right)^k \left(a_0\mathrm{e}^{-4\pi^2\alpha_0/\varepsilon} + a_1\mathrm{e}^{-4\pi^2\alpha_1/\varepsilon} + \cdots\right)$$
$$= \left(\frac{2\pi i}{\varepsilon}\right)^k a_0\mathrm{e}^{-4\pi^2\alpha_0/\varepsilon}\big(1 + o(\varepsilon^N)\big), \tag{2.2.3}$$

for all $N \geq 0$.

2.3 Dilogarithms

The dilogarithm is one of the simplest transcendental functions. It appears ubiquitously in conformal field theory, hyperbolic geometry and algebraic K-theory. In terms of Nahm's conjecture, the dilogarithm and variants of it appear naturally in the study of q-hypergeometric series. It is also used to detect torsion elements in the Bloch group of a number field, as seen in Section 2.4.

In this section we follow the exposition in [Lew81, Kir95, Zag07]. The dilogarithm is defined by the power series

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}, \quad |z| < 1.$$

We can analytically extend this to a function on $\mathbb{C} \setminus [1, \infty)$ via the integral representation

$$\mathrm{Li}_2(z) = -\int_0^z \log(1-u)\frac{\mathrm{d}u}{u}.$$

Example 2.3.1. For $0 < q < 1$ consider the q-shifted factorial

$$(q)_n = (1-q)(1-q^2)\cdots(1-q^n). \tag{2.3.1}$$

Taking the logarithm of both sides of (2.3.1) we obtain

$$\log(q)_n = \sum_{k=1}^{n} \log(1-q^k). \tag{2.3.2}$$

For sufficiently large n we can approximate the right hand side of (2.3.2) with the integral

$$\int_1^n \log(1-q^t)\mathrm{d}t = \frac{1}{\log q}\int_q^{q^n} \log(1-x)\frac{\mathrm{d}x}{x} = \frac{1}{\log q}\big(\mathrm{Li}_2(q) - \mathrm{Li}_2(q^n)\big).$$

We next define the Rogers dilogarithm function. This variant of the dilogarithm is defined on the interval $(0, 1)$ and given by

$$L(x) = \mathrm{Li}_2(x) + \frac{1}{2}\log(x)\log(1-x).$$

We can extend this to a monotone increasing continuous real-valued function on all of $\mathbb{R}$ by setting

$$L(x) = \begin{cases} -L\big(x/(x-1)\big) & x < 0 \\ 0 & x = 0 \\ \pi^2/6 & x = 1 \\ 2L(1) - L(1/x) & x > 1. \end{cases}$$

It is also real analytic on $\mathbb{R} \setminus \{0, 1\}$. For any $x, y \in [0, 1]$, the Rogers dilogarithm satisfies the functional equations

$$L(x) + L(1-x) = \frac{\pi^2}{6} = L(1) \quad \text{for} \quad x \in \mathbb{R}, \tag{2.3.3a}$$

$$L(x) + L(y) + L(1-xy) + L\left(\frac{1-y}{1-xy}\right) + L\left(\frac{1-x}{1-xy}\right) = \frac{\pi^2}{2} = 3L(1), \tag{2.3.3b}$$

for $x \geq 0$ and $y < 1$. The last variant of the dilogarithm we introduce is the Bloch–Wigner dilogarithm, defined by

$$D(z) = \mathrm{Im}\left(\mathrm{Li}_2(z)\right) + \arg(1-z)\log|z|, \tag{2.3.4}$$

and is real analytic on $\mathbb{C} \setminus \{0, 1\}$, but still continuous at $z = 0, 1$. Since $D(z) = -D(\overline{z})$ we have $D(z) = 0$ for all $z \in \mathbb{R}$. The Bloch–Wigner dilogarithm also satisfies the functional equations

$$D(x) + D(1 - x) = 0, \tag{2.3.5a}$$

$$D(x) + D\left(\frac{1}{x}\right) = 0, \tag{2.3.5b}$$

$$D(x) + D(y) + D(1 - xy) + D\left(\frac{1 - y}{1 - xy}\right) + D\left(\frac{1 - x}{1 - xy}\right) = 0. \tag{2.3.5c}$$

2.4 Bloch group

The Bloch group is a cohomology group that is related to algebraic K-theory, hyperbolic geometry and the polylogarithm. Given $A \in M_r(\mathbb{Q})$ positive definite and symmetric, Nahm's conjecture links torsion elements of Bloch groups determined by A to the existence of $(B, C) \in \mathbb{Q}^r \times \mathbb{Q}$ such that (A, B, C) is a modular triple. Here we introduce the definition of the Bloch group in enough generality to state the conjecture.

Let F denote a field and $F^\times = F \setminus \{0\}$. Let $\Lambda^2 F^\times$ denote the group of formal abelian sums of $x \wedge y$ with $x, y \in F^\times$ modulo the relations

1. $x \wedge x = 0$
2. $(x_1 x_2) \wedge y = x_1 \wedge y + x_2 \wedge y$.

Let $\mathbb{Z}[F]$ denote the set of formal abelian linear combinations of symbols $[x]$ where $x \in F$. Suppose $\partial : \mathbb{Z}[F^\times \setminus \{1\}] \to \Lambda^2 F^\times$ is a $\mathbb{Z}$-linear map defined by $\partial([x]) = x \wedge (1 - x)$. Let $A(F) = \ker \partial$ and $C(F)$ be the subgroup of $A(F)$ generated by elements of the form

$$[x] + \left[\frac{1}{x}\right] \tag{2.4.1a}$$

$$[x] + [1 - x] \tag{2.4.1b}$$

$$V(x, y) := [x] + [1 - xy] + [y] + \left[\frac{1 - y}{1 - xy}\right] + \left[\frac{1 - x}{1 - xy}\right]. \tag{2.4.1c}$$

Remark 2.4.1. Note that we may view the five-term relation (2.4.1c) as $\sum [x_i]$ where $\{x_i\}$ is a cyclic 5-tuple of numbers satisfying the recursion $1 - x_i = x_{i-1} x_{i+1}$.

Remark 2.4.2. The fact that $C(F)$ is a subgroup of $A(F)$ follows from

$$\partial\Big([x] + \left[\frac{1}{x}\right]\Big) = \partial\big([x] + [1 - x]\big) = \partial\big(V(x, y)\big) = 0.$$

This is seen in the following easy computations:

$$\begin{aligned}
\partial\Big([x] + \left[\frac{1}{x}\right]\Big) &= x \wedge (1 - x) + \frac{1}{x} \wedge \Big(1 - \frac{1}{x}\Big) \\
&= x \wedge (1 - x) + \frac{1}{x} \wedge \Big(\frac{x - 1}{x}\Big) \\
&= x \wedge (1 - x) - x \wedge (1 - x) \\
&= 0,
\end{aligned}$$

$$\begin{aligned}\partial\big([x]+[1-x]\big) &= x\wedge(1-x)+(1-x)\wedge x\\ &= 0,\end{aligned}$$

and

$$\partial\Big(\Big[\frac{1-x}{1-xy}\Big]\Big) = (1-y)\wedge y+(1-y)\wedge(1-x)+(1-y)\wedge\frac{1}{1-xy}+\Big(\frac{1}{1-xy}\Big)\wedge y + \Big(\frac{1}{1-xy}\Big)\wedge(1-x). \tag{2.4.2}$$

Interchanging x and y in (2.4.2) to obtain $\partial\big(\big[\frac{1-y}{1-xy}\big]\big)$, it is clear that $\partial\big(V(x,y)\big)=0$.

By convention we set $[0]=[1]=[\infty]=0$ in $A(F)$. The Bloch group is $\mathcal{B}(F):=A(F)/C(F)$. Notice that (2.4.1) describe the main functional equations for the Rogers and Bloch–Wigner dilogarithms given in (2.3.3) and (2.3.5) respectively.

We will now define a map $\mathcal{D}:\mathcal{B}(\mathbb{C})\to\mathbb{R}$ using the Bloch–Wigner dilogarithm. Setting $\xi=\sum_{i=1}^{n} n_i[z_i]\in\mathcal{B}(\mathbb{C})$ we have

$$\mathcal{D}(\xi)=\sum_{i=1}^{n} n_i D(z_i).$$

We are interested in the case when F is a number field or $\mathbb{C}$. Suppose $[F:\mathbb{Q}]=r_1+2r_2$ where r_1 denotes the number of real embeddings of F and r_2 denotes the number of complex conjugate non-real embeddings. Suppose the conjugate non-real embeddings are $\sigma_1,\dots,\sigma_{r_2}$. It is well known that $\xi\in\mathcal{B}(F)$ is a torsion element (element of finite order) if and only if it lies in the kernel of the map

$$\xi\mapsto\big(\mathcal{D}(\sigma_1(\xi)),\dots,\mathcal{D}(\sigma_{r_2}(\xi))\big).$$

By a result of Borel we have $\mathcal{B}(F)/\{\text{torsion}\}$ is isomorphic to $\mathbb{Z}^{r_2}$ [Bor80, Zag07].

Example 2.4.3. [Zag07] Consider the algebraic number

$$\alpha=\sqrt{\frac{-1+\sqrt{5}}{2}}.$$

Setting $F=\mathbb{Q}(\alpha)$, we define

$$\xi=[\alpha]+\Big[\frac{1}{1+\alpha}\Big]\in\mathbb{Z}[F].$$

Observe that $x=\alpha$ and $y=1/(1+\alpha)$ satisfy

$$1-x=x^4y \quad\text{and}\quad 1-y=xy.$$

The above equalities are used to verify $\xi\in\mathcal{B}(F)$ in the following computation,

$$\partial(\xi)=x\wedge\big(4x+y\big)+y\wedge\big(x+y\big)=0.$$

Moreover, ξ is a torsion element in $\mathcal{B}(F)$. Using the five term relation we compute

$$V(\alpha,\alpha)=2[\alpha]+2\Big[\frac{1-\alpha}{1-\alpha^2}\Big]+[1-\alpha^2]=2\xi+\Big[\frac{1-\sqrt{5}}{2}\Big].$$

We claim $\left[(1-\sqrt{5})/2\right]$ is 5-torsion in $\mathcal{B}(F)$. To see this, set $x_i = \beta$ in the recursion contained in Remark 2.4.1. We obtain the equation $1 - \beta = \beta^2$, so $\beta = (-1 \pm \sqrt{5})/2$. Thus

$$\left[\frac{1-\sqrt{5}}{2}\right] = -\left[\frac{-1+\sqrt{5}}{2}\right]$$

is 5-torsion in $\mathcal{B}\big(\mathbb{Q}(\sqrt{5})\big) \subset \mathcal{B}(F)$, and we see that ξ is indeed 10-torsion.

Chapter 3

Nahm's conjecture

3.1 Motivation and statement of Nahm's conjecture

Recall $q = \mathrm{e}^{2\pi i\tau}$ where $\tau \in \mathbb{H}$. Suppose $A = (a_{ij}) \in M_r(\mathbb{Q})$ is positive definite and symmetric, $B = (b_i) \in \mathbb{Q}^r$ and $C \in \mathbb{Q}$. Consider the Eulerian series

$$F_{A,B,C}(q) = \sum_{n\in(\mathbb{Z}_{\geq 0})^r} \frac{q^{\frac{1}{2}n^TAn+Bn+C}}{(q)_{n_1}\cdots(q)_{n_r}}, \tag{3.1.1}$$

which converges absolutely for $|q| < 1$. In 1994 Nahm has posed the problem of listing all modular triples (A, B, C) [Nah94, Nah95, Nah07]. He also conjectured conditions on the matrix A such that there exist some B and C such that (A, B, C) is a modular triple.

Let the general summand of (3.1.1) be denoted by $a_n(q) = a_{(n_1,\ldots,n_r)}(q)$. Suppose $q \to 1^-$ and $n_i \to \infty$ so that $q^{n_i} \to Q_i \notin \{0, 1\}$. Then

$$\frac{a_{n+e_i}(q)}{a_n(q)} = \frac{q^{(An)_i+b_i+\frac{1}{2}a_{ii}}}{1 - q^{n_i+1}} \to \frac{1}{1 - Q_i}\prod_{j=1}^{r} Q_j^{a_{ij}}. \tag{3.1.2}$$

The following well known lemma motivated by (3.1.2) is important to the study of Nahm's conjecture.

Lemma 3.1.1 ([VZ11])**.** Let A be as above. Then the system of equations

$$1 - Q_i = \prod_{j}^{r} Q_i^{a_{ij}}, \qquad i = 1, \ldots, r, \tag{3.1.3}$$

has a unique solution $Q^0 := (Q_1^0, \ldots, Q_r^0) \in (0, 1)^r$.

The starting point for Nahm's conjecture is to first search for the triples (A, B, C) such that $F_{A,B,C}(q)$ is a modular function with respect to any congruence subgroup Γ and weight k. In order for $F_{A,B,C}(q)$ to be modular we need the asymptotic expansion of $F_{A,B,C}(\mathrm{e}^{-\varepsilon})$ to match the form given in (2.2.1). Comparing (2.2.1) and (3.1.4) yields the following well known corollary.

Theorem 3.1.2 ([VZ11, Theorem 2.3]). There is an asymptotic expansion

$$F_{A,B,C}(\mathrm{e}^{-\varepsilon})\mathrm{e}^{-\alpha/\varepsilon} \sim \beta \mathrm{e}^{-\gamma\varepsilon}\left(1+\sum_{p=1}^{\infty} c_p\varepsilon^p\right), \quad \varepsilon \searrow 0 \tag{3.1.4}$$

with $\alpha \in \mathbb{R}_{>0}$ and $\beta, \gamma, c_p \in \overline{\mathbb{Q}}$, $p \geq 1$.

Remark 3.1.3 ([VZ11, Theorem 2.3]). One can be more explicit with the constants in (3.1.4). Writing

$$\tilde{A} = A + \operatorname{diag}\{\zeta_i\}_{1\leq i\leq n}, \quad \zeta_i = \frac{Q_i^0}{1-Q_i^0},$$

we have

$$\alpha = \sum_{i=1}^{r}\left((L(1)-L(Q_i^0)\right) > 0$$

$$\beta = \frac{1}{\sqrt{\det \tilde{A}}}\prod_{i=1}^{r}\frac{(Q_i^0)^{b_i}}{\sqrt{1-Q_i^0}}$$

$$\gamma = C + \frac{1}{24}\sum_{i=1}^{r}\frac{1+Q_i^0}{1-Q_i^0}.$$

Corollary 3.1.4 ([VZ11, Corollary 3.2]). If $F_{A,B,C}(q)$ is a modular function of weight k then

(a) it has weight $k=0$,

(b) $\alpha \in \pi^2\mathbb{Q}$ if and only if $\sum_{i=1}^{r} L(Q_i^0) \in \pi^2\mathbb{Q}$,

(c) $\mathrm{e}^{-\gamma\varepsilon}\left(1+\sum_{p=1}^{\infty} c_p\varepsilon^p\right) = 1$ if and only if $c_p = \frac{\gamma^p}{p!}$ for all p.

We will explain why the condition (b) in Corollary 3.1.4 is important in motivating Nahm's conjecture. Consider an arbitrary solution $Q = (Q_1,\dots,Q_r) \in \overline{\mathbb{Q}}^r$ to (3.1.3) and set $F = \mathbb{Q}(Q_1,\dots,Q_r)$. Set $\xi_Q = [Q_1]+\cdots+[Q_r] \in \mathbb{Z}[F^\times \setminus \{1\}]$. The following computation in $\Lambda^2(F^\times)$ proves that $\xi_Q \in \mathcal{B}(F)$:

$$\begin{aligned}\partial(\xi_Q) &= \sum_{i=1}^{r} Q_i \wedge (1-Q_i)\\ &= \sum_{i=1}^{r} Q_i \wedge \left(\prod_{j=1}^{r} Q_i^{a_{ij}}\right)\\ &= \sum_{i=1}^{r} Q_i \wedge \left(\sum_{j=1}^{r} a_{ij}Q_j\right)\\ &= \sum_{i=1}^{r}\sum_{j=1}^{r} a_{ij}(Q_i\wedge Q_j).\end{aligned} \tag{3.1.5}$$

Since $A=(a_{ij})$ is symmetric and $Q_i \wedge Q_j$ is antisymmetric in i and j, we see that (3.1.5) is 0 in $\Lambda^2(F^\times)$. Thus every solution to (3.1.3) defines an element in the Bloch group of the corresponding number field.

Since the exponents in (3.1.3) are rational, we need to be careful when determining powers $Q_i^{a_{ij}}$. For the determination of rational powers to be consistent we require that $Q_i = \mathrm{e}^{u_i}$ and $1 - Q_i = \mathrm{e}^{v_i}$ for some $u, v \in \mathbb{C}^r$ such that $v = Au$. This will define the minimal number field for which (3.1.1) is defined.

If (3.1.1) is modular then we must have $\sum_{i=1}^r L(Q_i^0) \in \pi^2\mathbb{Q}$. The Rogers dilogarithm takes values in $\pi^2\mathbb{Q}$ for combinations of real arguments of the form in (2.4.1a) and (2.4.1b). These are the main functional equations for $L(x)$. Thus it would be natural to expect that $\xi_{Q^0} = [Q_1^0] + \cdots + [Q_r^0]$ is a torsion element in $\mathcal{B}(F)$. This led Nahm to make the following conjecture.

Conjecture 3.1.5 (Nahm)**.** Given a positive definite symmetric $A \in M_r(\mathbb{Q})$, the following are equivalent:

(a) Every solution $Q = (Q_1, \ldots, Q_r)$ to (3.1.3), $\xi_Q = [Q_1] + \cdots + [Q_r] \in \mathcal{B}(F)$ is a torsion element, where $F = \mathbb{Q}(Q_1, \ldots, Q_r)$.

(b) There exist $(B, C) \in \mathbb{Q}^r \times \mathbb{Q}$ such that $F_{A,B,C}(q)$ is modular.

Let us now consider the implications of condition (c) in Corollary 3.1.4. Since $c_p \in \mathbb{Q}[B, \zeta, \tilde{A}^{-1}]$, we have infinitely many polynomial equations that the modular triple (A, B, C) must satisfy. These are given by

$$\left(c_p - \frac{1}{p!}c_1^p\right)(B, \zeta, \tilde{A}^{-1}) = 0 \quad \text{for} \quad p = 2, 3, \ldots$$

We refer the reader to [VZ11] for more details on this. It is crucial to point out that once A is fixed, so is ζ, and we obtain polynomials in $\mathbb{Q}(Q_1^0, \ldots, Q_r^0)[b_1, \ldots, b_r]$ of which the entries of modular B must be a root. We extensively use this observation in Section 4.9.

3.2 Rank one

Nahm's conjecture holds for $r = 1$, see [Zag07]. Moreover, Zagier and Terhoeven proved the following stronger result.

Theorem 3.2.1 ([NRT93, Ter94, Zag07])**.** Let $r = 1$ and view A as a scalar. The only $(A, B, C) \in \mathbb{Q}_+ \times \mathbb{Q} \times \mathbb{Q}$ for which $F_{A,B,C}(q)$ is modular are the ones given in the following table.

A	B	C	$F_{A,B,C}(q)$
2	0	$-1/60$	$\theta_{5,1}(\tau)/\eta(\tau)$
	1	$11/60$	$\theta_{5,2}(\tau)/\eta(\tau)$
1	0	$-1/48$	$\eta(\tau)^2/\eta(\tau/2)\eta(2\tau)$
	$1/2$	$1/24$	$\eta(2\tau)/\eta(\tau)$
	$-1/2$	$1/24$	$2\eta(2\tau)/\eta(\tau)$
$1/2$	0	$-1/40$	$\theta_{5,1}(\tau/4)/\theta_8(\tau)$
	$1/2$	$1/40$	$\theta_{5,2}(\tau/4)/\theta_8(\tau)$

Here,

$$\theta_8(\tau) = \sum_{n>0} \left(\frac{8}{n}\right) q^{n^2/8} = \frac{\eta(\tau)\eta(4\tau)}{\eta(2\tau)}$$
$$\theta_{5,j}(\tau) = \sum_{n\equiv 2j-1 \pmod{10}} (-1)^{\lfloor n/10 \rfloor} q^{n^2/40}.$$

Remark 3.2.2. The first two lines in the table are just the Rogers–Ramanujan identities as discussed in the introduction.

The main idea behind the proof of Theorem 3.2.1 is again a comparison of asymptotics. If $f(\tau) = F(q) \in M_k(\Gamma)$, then $g(\tau) = \tau^{-k} f\left(-\frac{1}{\tau}\right) \in M_k(S\Gamma S)$, and thus g has an expansion of the form occurring on the right hand side of (2.2.2). Taking the logarithm of the right hand side of (2.2.3) we obtain

$$\log F\left(e^{-\epsilon}\right) = -\frac{4\pi^2 n_0}{\varepsilon} - k\log\varepsilon + \log\left((2\pi i)^k a_0\right) + O(\varepsilon^N) \tag{3.2.1}$$

for all $N \geq 0$. Zagier compares (2.2.2) to the asymptotics of $\log F_{A,B,C}\left(e^{-\varepsilon}\right)$ as $\varepsilon \searrow 0$ and derives consequences (although for rank 1) similar to Corollary 3.1.4. See [Zag07] for more details.

3.3 Counterexamples

Nahm's conjecture is false for $r \geq 2$. Vlasenko and Zwegers in [VZ11] provide matrices $A^\star$ that do not satisfy condition (a) of Conjecture 3.1.5 but corresponding modular function $F_{A^\star,B^\star,C^\star}(q)$ exists [VZ11]. The main idea is to build modular triples $(A^\star, B^\star, C^\star)$ from modular triples (A, B, C) in such a way that the solutions of (3.1.3) corresponding to $A^\star$ do not satisfy condition (a) of Conjecture 3.1.5. For any $m \in \mathbb{Z}_{>0}$, the following result constructs a rank mr modular triple from one of rank r, see Theorem [VZ11, Theorem 4.2]. They apply this theorem to obtain

$$A = 1/2 \rightsquigarrow A^\star = \begin{pmatrix} 3/4 & -1/4 \\ -1/4 & 3/4 \end{pmatrix}.$$

The system of algebraic equations determined by this matrix has both torsion and non-torsion solutions in the Bloch group. However, it is still expected that if a modular triple (A, B, C) exists, then the solution $Q^0 \in (0,1)^r$ is torsion. The other direction of Nahm's conjecture is still open i.e. if the system of algebraic equations determined by general A specified in Conjecture 3.1.5 has only torsion solutions in the Bloch group, then this guarantees the existence of at least one modular triple (A, B, C).

3.4 Zagier's experiments

In [Zag07] Zagier experimented with modifying condition (a) of Conjecture 3.1.5 and proved that his modifications do not lead to a true statement. First he asked whether all solutions to (3.1.3) must be torsion? In particular, is it sufficient to require that just $\xi_{Q^0} = [Q_1^0] + \cdots + [Q_r^0] \in \mathcal{B}(\overline{\mathbb{Q}} \cap \mathbb{R})$ be torsion in $\mathcal{B}(\mathbb{C})$? To address this question, consider the matrix

$$A = \begin{pmatrix} 8 & 5 \\ 5 & 4 \end{pmatrix}.$$

In this case

$$Q^0 = \left(\phi^{-1}\psi, \phi^4 - \phi^3\psi\right)$$

where

$$\phi = \frac{\sqrt{5}+1}{2} \quad \text{and} \quad \psi = \frac{1+\sqrt{2\sqrt{5}-1}}{2}.$$

The element $\xi_{Q^0} = [\phi^{-1}\psi] + [\phi^4 - \phi^3\psi]$ is torsion since $\mathcal{L}(\xi) = \frac{8}{5}L(1)$ for both ξ_{Q^0} and its real conjugate, and $\mathcal{D}(\xi) = 0$ for both ξ_{Q^0} and its complex conjugate. However the equations

$$\begin{cases} 1 - Q_1 = Q_1^8 Q_2^5 \\ 1 - Q_2 = Q_1^5 Q_2^4 \end{cases}$$

have a Galois orbit of four solutions (Q_1, Q_2) that belong to a different quartic field where $\mathcal{D}(\xi) \neq 0$. His computer search yielded no $(B, C) \in \mathbb{Q}^2 \times \mathbb{Q}$ making $F_{A,B,C}(q)$ modular.

Zagier also showed that replacing condition (a) of Conjecture 3.1.5 with the stronger but more elementary requirement that all solutions to (3.1.3) must be real does not work. To see this, consider the matrix appearing on Zagier's list

$$A = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}.$$

The corresponding algebraic system of equations is

$$\begin{cases} 1 - Q_1 = Q_1^4 Q_2 \\ 1 - Q_2 = Q_1 Q_2. \end{cases}$$

This has two real and two complex solutions, all of which correspond to torsion elements in $\mathcal{B}(\mathbb{C})$. If the assumption is made that all solutions to (3.1.3) must be real then there should be no $(B, C) \in \mathbb{Q}^2 \times \mathbb{Q}$ which would result in $F_{A,B,C}(q)$ being modular. However,

$$F_{\left(\begin{smallmatrix} 4 & 1 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ -1/2 \end{smallmatrix}\right), 1/120}(q) = \frac{\theta_{5,1}(2\tau)}{\eta(\tau)} \quad \text{and} \quad F_{\left(\begin{smallmatrix} 4 & 1 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 \\ 1/2 \end{smallmatrix}\right), 1/120}(q) = \frac{\theta_{5,2}(2\tau)}{\eta(\tau)}.$$

So total reality is too strong an assumption.

Chapter 4

Representation theory and modularity

Let $A = C_X \otimes C_{X'}^{-1}$ where (X, X') is a pair of Dynkin diagrams of ADE or T type and C_X is the Cartan matrix of X. For each matrix in this family it has been conjectured that there is at least one B and C such that (A, B, C) is a modular triple [Lee13]. There are several specialised conjectures and partial results in the literature related to this problem. Modularity up to a rational power of q is conjectured for the generalized Andrews–Gordon series $\chi_{k,l}(q)$ defined in [BCFK14, Conjecture 4.1, Section 4.5] i.e. when $A = C_{\mathrm{A}_{n-1}} \otimes C_{\mathrm{T}_{k-1}}^{-1}$. The B-vector occurring in $\chi_{k,l}(q)$ is determined by the value of $1 \leq l \leq k$. Thus to prove the above conjecture when $A = C_{\mathrm{A}_{n-1}} \otimes C_{\mathrm{T}_{k-1}}^{-1}$, it suffices to produce for each $n, k \geq 2$ at least one value of l for which $\chi_{k,l}(q)$ is modular up to a rational power of q. Warnaar and Zudilin originally studied $\chi_{k,l}(q)$ in relation to A_{n-1} root systems [WZ12]. Modularity of $\chi_{2,1}(q)$ when n is odd follows from an identity in their paper [WZ12, Theorem 2.3]. They have conjectured a more general identity for $\chi_{k,1}(q)$ when n is odd [WZ12, Conjecture 2.2], inspired by the Feigin-Stoyanovsky Theorem, an identity $\chi_{k,k}(q)$ when n odd [FS94, Sto98]. Subsequently Bringmann et al. studied the modular properties of the graded dimensions of principal subspaces of level one $A_{n-1}^{(1)}$-modules in [BCFK14]. Modularity of $\chi_{k,k}(q)$ when n is odd and $\chi_{2,l}(q)$ when n is even follows from their work, see [BCFK14, Theorem 1.1].

In this chapter we consider modularity conjecture in [Lee13] when $A = C_X \otimes C_{A_{k-1}}^{-1}$, where X is Dynkin diagram of ADE type. Let $C_{\mathrm{A}_{n-1}} := (C_{uv})_{1 \leq u,v \leq n-1}$ and $C_{\mathrm{A}_{k-1}} := (C_{st})_{1 \leq s,t \leq k-1}$. We prove the following proposition in Section 4.10.

Proposition 4.0.1. Suppose $\gcd((n-1)!, k) = 1$. Choosing $(k_0, j) \in \{1, 2, \ldots, k-1\} \times \{1, \ldots, n-1\}$ and setting $k_j = k - k_0$, the following series
(4.0.1)
$$q^{T(n,k,k_0,j)} \sum_{(m_i^{(s)}) \in M_{(k-1)\times(n-1)}(\mathbb{Z}_{\geq 0})} \frac{q^{\frac{1}{2}\sum \left(C_{\mathrm{A}_{n-1}}\right)_{uv} \left(C_{\mathrm{A}_{k-1}}^{-1}\right)_{st} m_u^{(s)} m_v^{(t)}}}{\prod_{i=1}^{n-1} \prod_{s=1}^{k-1} (q)_{m_i^{(s)}}} q^{\sum_{s=k_0+1}^{k-1} (s-k_0) m_j^{(s)} - \frac{k_j}{k} \sum_{s=1}^{k-1} s m_j^{(s)}},$$

is a modular function for some $T(n, k, k_0, j) \in \mathbb{Q}$ with respect to $\Gamma(s)$, where $s = \mathrm{lcm}\{k, n, n+k, n-1\}$.

In terms of Nahm language, it is clear that $A = C_{\mathrm{A}_{n-1}} \otimes C^{-1}_{\mathrm{A}_{k-1}}$ in (4.0.1). The modular B-vectors in (4.0.1) have a beautiful structure. Viewing $\left(m_i^{(s)}\right) \in M_{(k-1)\times(n-1)}(\mathbb{Z}_{\geq 0})$, the modular B-"vectors" in $M_{(k-1)\times(n-1)}(\mathbb{Z}_{\geq 0})$ have jth column as $-C^{-1}_{\mathrm{A}_{k-1}} e_{k_0}$ and 0s elsewhere. Setting $n = 2$ in Proposition 4.0.1 proves a claim of Keegan and Nahm in [KN11]. This case has stong ties to conformal field theory and was studied experimentally in [KN11]. In particular, Keegan and Nahm considered the case when B is any column vector of $-C^{-1}_{\mathrm{A}_{k-1}}$. For $k = 2, \ldots, 12$, they decomposed the corresponding Nahm type sum into coset characters in order to establish modularity, but higher level cases were not considered in their paper.

The main idea behind the proof Proposition 4.0.1 is to write (4.0.1) as a sum of string functions arising from the irreducible $\mathrm{A}^{(1)}_{n-1}$-module of level k, with highest weight $\Lambda = k_0\Lambda_0 + k_j\Lambda_j$, $k_0 + k_j = k \geq 2$. There are two main ingredients. The first being that string functions of an affine Kac–Moody algebra $\mathfrak{g}$ of rank $l+1$ are modular forms of weight $-\frac{1}{2}l$, see Theorem 4.6.3. The second, a formula due to Georgiev for the string functions of an irreducible $\mathrm{A}^{(1)}_{n-1}$-module of highest weight Λ taking the form above [Geo94]. The formula of Georgiev is packaged as a Nahm-type sum of the form featured in (4.9) with the summation over a system of congruences involving the columns of $\left(m_i^{(s)}\right)$ modulo k. The proof mainly invloves checking there is sufficiently many string functions to sum in order to hit every congruence class modulo k and hence obtain (4.0.1), see Section 4.10 for more details.

We conjecture the trick used in the proof of Proposition 4.0.1 will generalise to proving the modularity when X is a Dynkin diagram of DE type and B the trivial vector. Hatayama et al. in [HKK+99, Conjecture 6.1] have conjectural formulae for the string functions coming from tensor product of vacuum modules of an arbitrary non-twisted affine Lie algebra $X^{(1)}_n$. These formulae are of Nahm-type, similar to the formula of Georgiev (4.10.1).

Before proceeding to the work mentioned above, we first develop some basics on the representation theory of affine Kac–Moody algebras. From this we are able to access characters, string functions and their modularity properties. At some points we will discuss topics in the context of general Kac–Moody algebras, we will be explicit when this happens.

In this chapter we also investigate the modular B-vectors of the classical Andrews–Gordon series i.e. $\chi_{k,l}(q)$ when $n = 2$, or see the left hand side of (4.9.1). One may establish modularity for the simple family of B-vectors corresponding to $l = 1, \ldots, k$ by comparing the series to characters arising from the $M(2, 2k+1)$-minimal model [BPZ84, DFMS97]. Keegan and Nahm ask whether these are the only modular B-vectors? For small values of k, they carry out a computational search in [KN11] for all modular B-vectors within the range $-8 \leq b_i \leq 8$. Only vectors in the simple family were obtained. We are able to confirm that this phenomenon happens with no restrictions on the b_i when $k = 3$ and $k = 4$ in Section 4.9 using `Magma` and the asymptotics developed in Section 3.1. A brief note will be made on the Virasoro algebra, minimal models and their characters in order to present this work.

4.1 Preliminaries for affine Kac–Moody algebras

We follow the notation set out in [Kac90, Chap 6]. Let A be a generalized Cartan matrix (GCM) of affine type having rank l. Let $a = (a_0, a_1, \dots, a_l) \in (\mathbb{Z}_{>0})^{l+1}$ be the unique vector such that $Aa = 0$ with $\gcd(a_0, a_1, \dots, a_l) = 1$. The sequence $a_0, a_1, \dots, a_l$ are the Dynkin labels of the Dynkin diagram $S(A)$ associated to A. We have $a_0 = 1$ unless $A = \mathrm{A}_{2l}^{(2)}$, in which case $a_0 = 2$. Denote by $a_i^\vee$ the labels of the Dynkin diagram $S(A^t)$ of the dual algebra. This is obtained from $S(A)$ be reversing all of the arrows and preserving the labelling of the vertices. In all cases we have $a_0^\vee = 1$. Let

$$h := \sum_{i=0}^{l} a_i \quad \text{and} \quad h^\vee := \sum_{i=0}^{l} a_i^\vee,$$

denote the Coxeter and dual Coxeter numbers respectively.

Let $\mathfrak{g} := \mathfrak{g}(A)$ be the Kac–Moody algebra associated to affine GCM $A = (a_{ij})$ (columns and rows indexed by the labels $0, 1, \dots, l$). Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{h}^*$ the dual of $\mathfrak{h}$. Suppose $\mathfrak{g}$ has generators e_i and f_i, $i = 0, \dots, l$, satisfying the Serre relations

$$\begin{cases} [e_i, f_i] = \delta_{ij}\alpha_i^\vee & \text{if } i, j = 0, 1, \dots, l \\ [h, h'] = 0 & \text{if } h, h' \in \mathfrak{h} \\ [h, e_i] = \langle \alpha_i, h\rangle e_i & \text{if } h \in \mathfrak{h} \\ [h, f_i] = -\langle \alpha_i, h\rangle & \text{if } h \in \mathfrak{h} \\ \mathrm{ad}(e_i)^{1-a_{ij}}(e_j) = 0 & \text{if } i, j = 0, 1, \dots, l \\ \mathrm{ad}(f_i)^{1-a_{ij}}(f_j) = 0 & \text{if } i, j = 0, 1, \dots, l. \end{cases}$$

where $\Pi^\vee = \{\alpha_0^\vee, \dots, \alpha_l^\vee\} \subset \mathfrak{h}$ and $\Pi = \{\alpha_0, \dots, \alpha_l\} \subset \mathfrak{h}^*$ are the sets of simple coroots and simple roots respectively.

By [Kac90, Lem. 4.6], any matrix A of affine type is symmetrizable. Moreover

$$A = \mathrm{diag}\,(a_0/a_0^\vee, \dots, a_l/a_l^\vee)\, B, \tag{4.1.1}$$

where B is a symmetric matrix.

Fix the scaling element $d \in \mathfrak{h}$ such that

$$\langle \alpha_0, d\rangle = 1 \quad \text{and} \quad \langle \alpha_i, d\rangle = 0 \quad \text{for} \quad i = 1, \dots, l.$$

The $\alpha_0^\vee, \dots, \alpha_l^\vee, d$ form a basis for $\mathfrak{h}$. We define a non-degenerate symmetric $\mathbb{C}$-bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}$ by

$$\begin{cases} (\alpha_i^\vee|\alpha_j^\vee) = (a_j/a_j^\vee)a_{ij} & \text{if } i, j = 0, 1, \dots, l \\ (\alpha_i^\vee|d) = 0 & \text{if } i = 1, \dots, l \\ (\alpha_0^\vee|d) = a_0 \\ (d|d) = 0. \end{cases} \tag{4.1.2}$$

This can be extended uniquely to a non-degenerate symmetric bilinear form on all of $\mathfrak{g}$, called the normalized invariant form.

We now describe the induced $\mathbb{C}$-bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}^*$. First define an element $\Lambda_0 \in \mathfrak{h}^*$ by

$$\langle \Lambda_0, d\rangle = 0 \quad \text{and} \quad \langle \Lambda_0, \alpha_i^\vee\rangle = \delta_{0i} \quad \text{for} \quad i = 0, 1, \ldots, l.$$

Then $\alpha_0, \ldots, \alpha_l, \Lambda_0$ is a basis for $\mathfrak{h}^*$ and the bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}$ induces an isomorphism $\nu : \mathfrak{h} \to \mathfrak{h}^*$. The isomorphism has relations

$$\nu\left(\alpha_i^\vee\right) = \left(a_i/a_i^\vee\right)\alpha_i, \quad \nu(c) = \delta \quad \text{and} \quad \nu(d) = a_0\Lambda_0,$$

where

$$\delta = \sum_{i=0}^{\infty} a_i\alpha_i$$

is the null root. Moreover the form on $(\cdot|\cdot)$ on $\mathfrak{h}^*$ can be described by

$$\text{(4.1.3)} \qquad \begin{cases} (\alpha_i|\alpha_i) = a_i^{-1}a_i^\vee a_{ij} & \text{if } i, j = 0, 1, \ldots, l \\ (\alpha_i|\Lambda_0) = 0 & \text{if } i = 1, \ldots, l \\ (\alpha_0|\Lambda_0) = a_0^{-1} & \\ (\Lambda_0|\Lambda_0) = 0. & \end{cases}$$

The forms defined by (4.1.2) and (4.1.3) can be used to deduce the following useful formulae

$$\begin{aligned} &(\delta|\Lambda_0) = 1 \qquad (\delta|\delta) = 0 \qquad (\delta|\alpha_i) = 0, \quad \text{for} \quad i = 0, 1, \ldots, l, \\ &(c|\alpha_0) = a_0 \qquad (c|c) = 0 \qquad (c|\alpha_i^\vee) = 0, \quad \text{for} \quad i = 0, 1, \ldots, l, \end{aligned}$$

where

$$c := \sum_{i=0}^{l} a_i^\vee \alpha_i^\vee$$

is the canonical central element.

Let $\mathring{\mathfrak{g}}$ be the classical part of $\mathfrak{g}$, $\mathring{\mathfrak{h}}$ its Cartan subalgebra and $\mathring{\mathfrak{h}}^*$ the dual of $\mathfrak{h}$. Noticing $\mathring{\mathfrak{h}} = \mathrm{span}_{\mathbb{C}}\{\alpha_1^\vee, \ldots, \alpha_l^\vee\}$ and $\mathring{\mathfrak{h}}^* = \mathrm{span}_{\mathbb{C}}\{\alpha_1, \ldots, \alpha_l\}$, we have the orthogonal direct sum decompositions

$$\text{(4.1.4)} \qquad \mathfrak{h} = \mathring{\mathfrak{h}} \oplus (\mathbb{C}c + \mathbb{C}d) \quad \text{and} \quad \mathfrak{h}^* = \mathring{\mathfrak{h}}^* \oplus (\mathbb{C}\delta + \mathbb{C}\Lambda_0)\,.$$

With (4.1.4) in mind, let $\overline{\lambda} \in \mathring{\mathfrak{h}}^*$ denote the orthogonal projection of $\lambda \in \mathfrak{h}^*$. Writing

$$\lambda - \overline{\lambda} = u_1\Lambda_0 + u_2\delta$$

for some $u_1, u_2 \in \mathbb{C}$, we compute the coefficients u_1 and u_2 in terms of λ. Thus

$$\begin{aligned} u_1 &= \langle \lambda - \overline{\lambda}|\delta\rangle = \langle \lambda|\delta\rangle - \langle \overline{\lambda}|\delta\rangle = \langle \lambda, c\rangle \\ u_2 &= \langle \lambda - \overline{\lambda}|\Lambda_0\rangle = \langle \lambda|\Lambda_0\rangle - \langle \overline{\lambda}|\Lambda_0\rangle = \langle \lambda|\Lambda_0\rangle. \end{aligned}$$

This results in the useful projection formula

$$\text{(4.1.5)} \qquad \lambda = \overline{\lambda} + \langle \lambda, c\rangle\Lambda_0 + \langle \lambda|\Lambda_0\rangle\delta.$$

4.2 Roots

Let Δ be the root system attached to the affine Kac–Moody algebra $\mathfrak{g}$ and $\mathring{\Delta}$ the root system of $\mathring{\mathfrak{g}}$. The root system controls the structure of $\mathfrak{g}(A)$ through its root space decomposition. Suppose $\mathfrak{g}(A)$ has such a decomposition, given by

$$\mathfrak{g}(A) = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \right),$$

where

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}.$$

We give an explicit description of the root system Δ when A is non-twisted. We also give an explicit description of the root spaces $\mathfrak{g}_\alpha$ in the loop realisation $\hat{\mathcal{L}}(\mathring{\mathfrak{g}})$.

Definition 4.2.1. If $\beta \in \Delta$ and β is of the form $\sum_{i=0}^{l} \mathbb{Z}_{\geq 0}\alpha_i$, then β is called a positive root. Let Δ_+ denote the set of positive roots.

Definition 4.2.2. A root $\alpha \in \Delta$ is called real if it is in the W-orbit (W is the Weyl group defined in Section 4.3) of the simple roots of $\mathfrak{g}$. Let Δ^{re} and Δ^{re}_+ denote the set of real and real positive roots respectively.

Definition 4.2.3. A root which is not real is called imaginary. Let Δ^{im} denote the set of imaginary roots.

Thus $\Delta^{\mathrm{re}} \cap \Delta^{\mathrm{im}} = \varnothing$ and $\Delta = \Delta^{\mathrm{re}} \cup \Delta^{\mathrm{im}}$. We have the following existence theorem for imaginary roots.

Theorem 4.2.4 ([Kac90, Theorem 5.6])**.** Let A be an indecomposable GCM.

(a) If A is of finite type then $\Delta^{\mathrm{im}} = \varnothing$.

(b) If A has affine type, then

$$\Delta^{\mathrm{im}} = \{k\delta : k \in \mathbb{Z} \setminus \{0\}\} \quad \Delta^{\mathrm{im}}_+ = \{k\delta : k \in \mathbb{Z}_{>0}\}.$$

The next proposition describes Δ^{re}_+ and Δ^{re}_+ in terms of $\mathring{\Delta}$ and δ in the non-twisted affine case.

Proposition 4.2.5 ([Kac90, Proposition 6.3])**.** Let $\mathfrak{g}(A)$ be a Lie algebra associated to a matrix of type $A = X_l^{(1)}$. Then

$$\Delta^{\mathrm{re}} = \{\alpha + n\delta | \alpha \in \mathring{\Delta}, n \in \mathbb{Z}\}$$

and

$$\Delta^{\mathrm{re}}_+ = \{\alpha \in \Delta_{\mathrm{re}} | n > 0\} \cup \mathring{\Delta}^{\mathrm{re}}_+.$$

The loop realisation $\hat{\mathcal{L}}(\mathring{\mathfrak{g}})$ is a concrete way of constructing the affine Kac–Moody algebra $\mathfrak{g}(A)$. We refer the reader [Kac90, Chapter 7] for more details on this. Considering an explicit description of the root spaces of $\hat{\mathcal{L}}(\mathring{\mathfrak{g}})$ gives a useful corollary on the dimension of the imaginary root spaces.

Example 4.2.6 ([Kac90, Section 7.4])**.** We give the root space decomposition of $\hat{\mathcal{L}}(\mathring{\mathfrak{g}})$ with respect to $\mathfrak{h}$. By Proposition 4.2.5,

$$\Delta = \{\alpha + n\delta : n \in \mathbb{Z}, \alpha \in \mathring{\Delta}\} \cup \{n\delta : j \in \mathbb{Z} \setminus \{0\}\}.$$

Writing

$$\hat{\mathcal{L}}(\mathring{\mathfrak{g}}) = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta} \mathcal{L}(\mathring{\mathfrak{g}})_\alpha \right),$$

the root spaces are given by

$$\mathcal{L}(\mathring{\mathfrak{g}})_{\alpha+n\delta} = t^n \otimes \mathring{\mathfrak{g}}_\alpha \quad \mathcal{L}(\mathring{\mathfrak{g}})_{n\delta} = t^n \otimes \mathring{\mathfrak{h}}.$$

Corollary 4.2.7 ([Kac90, Corollary 7.4])**.** Let $\mathfrak{g}(A)$ be a non-twisted Lie algebra of rank $l+1$. Then the multiplicity of every imaginary root of $\mathfrak{g}(A)$ is l.

4.3 Affine Weyl group

The affine Weyl group of an affine Kac–Moody algebra can be used to describe much of the structure of the algebra and its highest weight representations. For this see Sections 4.2, 4.5 and 4.6. We give details on the affine Weyl group and its properties in this section that we use later in this thesis.

For each integer $0 \le i \le l$ we introduce the fundamental reflection r_i on $\mathfrak{h}^*$ defined by

$$r_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i \quad \text{for} \quad \lambda \in \mathfrak{h}^*.$$

Each r_i is a reflection in the hyperplane orthogonal to the simple root α_i since its fixed-point set is

$$T_i = \{\lambda \in \mathfrak{h}^* | \langle \lambda, \alpha_i \rangle = 0\},$$

and $r_i(\alpha_i) = -\alpha_i$.

Definition 4.3.1. The subgroup of $\mathrm{GL}(\mathfrak{h}^*)$ generated by the r_i is called the affine Weyl group of A or $\mathfrak{g}(A)$, denoted W.

The subgroup of W generated by r_i for $1 \le i \le l$, denoted $\mathring{W}$, is the finite Weyl group attached to $\mathring{\mathfrak{g}}$. The group W has a simple description in terms of $\mathring{W}$. To give this description we first need to define a special lattice in $\mathring{\mathfrak{h}}^*$ that parametrizes a family of endomorphisms of $\mathfrak{h}^*$.

Remark 4.3.2. Note that if $\alpha \in \Delta^{\mathrm{re}}$, then $\alpha = w(\alpha_i)$ for some i and the reflection $r_\alpha = w r_i w^{-1} \in W$. Explicitly,

$$r_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha \quad \text{for} \quad \lambda \in \mathfrak{h}^*.$$

Lemma 4.3.3 ([Kac90, Lemma 6.5])**.** Let $\alpha \in \Delta_+^{\mathrm{re}}$ such that $\beta := \delta - a\alpha \in \Delta_+^{\mathrm{re}}$ for some a. Then

$$r_\alpha r_\beta(\lambda) = \lambda + \langle\lambda, c\rangle \nu(\beta^\vee) - \left(\langle\lambda, \beta^\vee\rangle + \frac{1}{2}|\beta^\vee|^2\langle\lambda, c\rangle\right)\delta,$$

for $\lambda \in \mathfrak{h}^*$.

Specialising Lemma 4.3.3 to $\alpha = \alpha_0$ and $\beta = \theta := \sum_{i=1}^{l} a_i\alpha_i$, we obtain the equation

$$r_{\alpha_0} r_\theta(\lambda) = \lambda + \langle\lambda, c\rangle \nu(\theta^\vee) - \left(\langle\lambda, \theta^\vee\rangle + \frac{1}{2}|\theta^\vee|^2\langle\lambda, c\rangle\right)\delta. \tag{4.3.1}$$

Consider the family of endomorphisms $t_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ for each $\alpha \in \mathring{\mathfrak{h}}^*$ given by

$$t_\alpha(\lambda) = \lambda + \langle\lambda, c\rangle\alpha - \left((\lambda|\alpha) + \frac{1}{2}|\alpha|^2\langle\lambda, c\rangle\right)\delta. \tag{4.3.2}$$

It is an elementary computation to verify additivity and normality properties of the t_α in the following lemma.

Lemma 4.3.4. The following hold:

(a) $t_\alpha t_\beta = t_{\alpha+\beta}$ for $\alpha, \beta \in \mathring{\mathfrak{h}}^*$

(b) $w t_\alpha w^{-1} = t_{w(\alpha)}$ for $w \in \mathring{W}$ and $\alpha \in \mathring{\mathfrak{h}}^*$.

After inspecting (4.3.1), consider the lattice $M := \nu\big(\mathbb{Z}[\mathring{W} \cdot \theta^\vee]\big) \subset \mathring{\mathfrak{h}}_{\mathbb{R}}^*$. In particular,

$$M = \begin{cases} \mathring{Q} & \text{if } A \text{ is symmetric or } r > a_0 \\ \nu\big(\mathring{Q}^\vee\big) & \text{otherwise,} \end{cases}$$

where

$$Q := \sum_{i=0}^{l} \mathbb{Z}\alpha_i \quad \text{and} \quad \mathring{Q} := \sum_{i=1}^{l} \mathbb{Z}\alpha_i.$$

Note Q is referred to as the root lattice and $\mathring{Q}$ as the classical root lattice.

By formula (4.3.2), M considered as an abelian group acts faithfully on $\mathfrak{h}^*$. Denote the corresponding subgroup in $\mathrm{GL}(\mathfrak{h}^*)$ by T. We call this the group of translations. Any element $w \in W$ can be uniquely written as the composition of a translation from T and a reflection from $\mathring{W}$ as seen in the next proposition.

Proposition 4.3.5 ([Kac90, Proposition 6.5])**.** We have $W = \mathring{W} \ltimes T$.

Remark 4.3.6. The semi-product structure of W will be employed primarily to simplify characters in Section 4.5.

The Weyl group W can be used to build an alcove geometry in $\mathring{\mathfrak{h}}^*_{\mathbb{R}}$ that is necessary to understand the structure of the weights in highest weight representations of affine Kac–Moody algebras. For each $\alpha \in \mathring{\Delta}$ and $k \in \mathbb{Z}$, define the following family of hyperplanes in $\mathring{\mathfrak{h}}^*_{\mathbb{R}}$

$$L_{\alpha,k} := \{\lambda \in \mathring{\mathfrak{h}}^*_{\mathbb{R}} : (\lambda|\alpha) = k\},$$

and the hyperplane

$$L_{\theta,1} := \{\lambda \in \mathring{\mathfrak{h}}^*_{\mathbb{R}} : (\lambda|\theta) = 1\}.$$

The generators $r_0, r_1, \ldots, r_l$ of W act on $\mathring{\mathfrak{h}}^*_{\mathbb{R}}$ as reflections in the hyperplanes $L_{\theta,1}, L_{\alpha_1,0}, \ldots, L_{\alpha_l,0}$ respectively. We will introduce a collection of affine hyperplanes, Ω, whose corresponding affine reflections will lie in W by Proposition 4.2.5 and Remark 4.3.2. When $\mathfrak{g}$ is non-twisted we have the simple description

$$\Omega = \{L_{\alpha,k} : \alpha \in \mathring{\Delta}, k \in \mathbb{Z}\}.$$

It turns out that W permutes the affine hyperplanes of Ω. Since W is generated by $r_\theta, r_1, \ldots, r_l$, this is sufficient to note the following lemma.

Lemma 4.3.7 ([Car05, Lemma 17.29])**.** The following hold:

(a) $r_i\left(L_{\alpha,k}\right) = L_{r_i(\alpha),k}$ for $i = 1, \ldots, l$,

(b) $r_\theta\left(L_{\alpha,k}\right) = L_{r_0(\alpha),k+(\alpha|\theta)}$,

(c) if $L_{\alpha,k} \in \Omega$ then $L_{r_0(\alpha),k+(\alpha|\theta)} \in \Omega$.

Naturally this leads to a group action of W on the set of *alcoves*.

Definition 4.3.8. The connected components of

$$\mathring{\mathfrak{h}}^*_{\mathbb{R}} - \bigcup_{L_{\alpha,k}\in\Omega} L_{\alpha,k}$$

are called alcoves. Let $\mathcal{A}$ be the set of alcoves.

Proposition 4.3.9 ([Car05, Proposition 17.28])**.** The set

$$C := \{\lambda \in \mathring{\mathfrak{h}}^*_{\mathbb{R}} : (\lambda|\alpha_i) > 0 \text{ for } i = 1, \ldots, l \text{ and } (\lambda|\theta) < 1\}$$

is an alcove.

The fundamental region for the action of W on $\mathring{\mathfrak{h}}^*_{\mathbb{R}}$ is called the affine alcove. The affine alcove is important because it can be used to compute the maximal dominant weights, and hence all strings of weights for any affine Kac–Moody algebra.

Definition 4.3.10. The affine alcove is defined as

$$C_{\text{af}} := \overline{C} = \{\lambda \in \mathring{\mathfrak{h}}^*_{\mathbb{R}} : (\lambda|\alpha_i) \geq 0 \text{ for } i = 1, \ldots, l \text{ and } (\lambda|\theta) \leq 1\}.$$

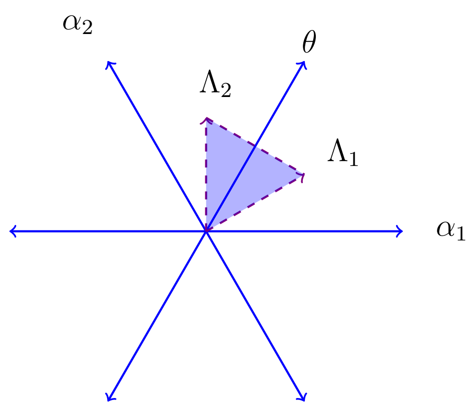

Figure 4.1: The shaded region is the alcove C for $A_2^{(1)}$.

4.4 Irreducible modules for affine algebras

We introduce the basic representation theory of affine Kac–Moody algebras. We must restrict our attention to a certain category of representations (or modules) with a well-defined character. These are precisely the representations that are $\mathfrak{h}$-diagonalisable with finite dimensional weight spaces.

We start by considering any Kac–Moody algebra $\mathfrak{g}$ of rank n. We have the usual triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+, \tag{4.4.1}$$

where $\mathfrak{n}_+$ and $\mathfrak{n}_-$ are generated by the Serre generators e_i and f_i respectively for $i = 1, \ldots, n$. The corresponding decomposition of the universal enveloping algebra

$$U(\mathfrak{g}) = U(\mathfrak{n}_-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_+). \tag{4.4.2}$$

Applying the Poincaré–Birkhoff–Witt Theorem [Mil13, Theorem 1.30], a basis for $U(\mathfrak{n}_-)$ is the set of monomials $e_1^{r_1} \cdots e_n^{r_n}$ where $r_i \in \mathbb{Z}_{\geq 0}$. Similarly $f_1^{m_1} \cdots f_n^{m_n}$ is a basis for $U(\mathfrak{n}_+)$ where $m_i \in \mathbb{Z}_{\geq 0}$.

Definition 4.4.1 (BGG category $\mathcal{O}$)**.** The objects in category $\mathcal{O}$ are the $\mathfrak{g}$-modules V that satisfy

(a) $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$ where $V_\lambda := \{v \in V : h \cdot v = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}$,

(b) $\dim V_\lambda < \infty$,

(c) There exists a finite set $\{\lambda_1, \ldots, \lambda_s\}$ such that for each λ with $V_\lambda \neq 0$ satisfies $\lambda \prec \lambda_i$ for some $i \in \{1, \ldots, s\}$, where $\prec$ is the partial order on $\mathfrak{h}^*$ defined below.

Definition 4.4.2. For $\lambda, \mu \in \mathfrak{h}^*$, $\mu \prec \lambda$ if and only if $\lambda - \mu$ is a non-negative linear combination of simple roots.

Definition 4.4.3. Any $\lambda \in \mathfrak{h}^*$ with $V_\lambda \neq 0$ is called a weight of V. Let $P(V)$ denotes the set of weights of V.

The morphisms in $\mathcal{O}$ are homomorphisms of $\mathfrak{g}$-modules. Any submodule or quotient of a module from $\mathcal{O}$ is another module in $\mathcal{O}$. A direct sum or tensor product of a finite number of modules from $\mathcal{O}$ is another module in $\mathcal{O}$.

We now explain how to obtain the $\mathfrak{g}$-module whose character is of interest in Section 4.5 and allows us to introduce string functions in Section 4.6. A starting point is a class of $\mathfrak{g}$-modules in category $\mathcal{O}$ called highest weight modules.

Definition 4.4.4. A $\mathfrak{g}$-module V is called a highest weight module of highest weight $\Lambda \in \mathfrak{h}^*$ if there exists a non-zero vector $v_\Lambda \in V$ (called the highest weight vector) such that

(a) $\mathfrak{n}_+(v_\Lambda) = 0$,

(b) $h \cdot v_\Lambda = \lambda(h) v_\Lambda$ for all $h \in \mathfrak{h}$,

(c) $U(\mathfrak{g})(v_\Lambda) = V$.

In light of conditions (a) and (b) in Definition 4.4.4, we may replace condition (c) with

$$U(\mathfrak{n}_-)(v_\Lambda) = V.$$

Thus V is spanned by vectors of the form $f_1^{m_1} \cdots f_n^{m_n} v_\Lambda$ where $m_i \in \mathbb{Z}_{\geq 0}$ and for $h \in \mathfrak{h}$

$$h(f_1^{m_1} \cdots f_n^{m_n} v_\Lambda) = (m_1\alpha_1 - \cdots - m_n\alpha_n)(h) f_1^{m_1} \cdots f_n^{m_n} v_\Lambda.$$

Thus it follows that

$$P(V) \subset \left\{ \Lambda - \sum_{i=1}^{n} m_i \alpha_i : i = 1, \ldots, n \right\}, \tag{4.4.3}$$

and moreover

$$V = \bigoplus_{\lambda \leq \Lambda} V_\lambda, \quad V_\Lambda = \mathbb{C} v_\Lambda \quad \text{and} \quad \dim V_\lambda < \infty.$$

This will be important when discussing characters of highest weight modules in Section 4.5.

Definition 4.4.5. A $\mathfrak{g}$-module $M(\Lambda)$ of highest weight Λ is called a Verma module if every $\mathfrak{g}$-module of highest weight Λ is a quotient of $M(\Lambda)$.

We have the following existence and uniqueness theorem for Verma modules.

Proposition 4.4.6 ([Kac90, Proposition 9.2])**.** For any Kac–Moody algebra $\mathfrak{g}$ the following hold:

(a) For every $\Lambda \in \mathfrak{h}^*$ there exists a unique up to isomorphism Verma module $M(\Lambda)$.

(b) $M(\Lambda)$ contains a unique proper maximal submodule $M'(\Lambda)$.

Thus it is clear from Proposition 4.4.6 that among $\mathfrak{g}$-modules of highest weight $\Lambda \in \mathfrak{h}^*$, there is a unique irreducible one

$$L(\Lambda) := M(\Lambda)/M'(\Lambda).$$

The irreducible $\mathfrak{g}$-module $L(\Lambda)$ is the one interest for the rest of this thesis.

4.5 Characters of affine algebras and string functions

We will now introduce four important lattices P, P_+, P_{++} and $\mathring{P}$ needed to study characters of representations of an affine Kac–Moody algebra $\mathfrak{g}(A)$. These lattices, referred to as the weight lattice, the set of integral weights, set of of dominant weights respectively and classical weight lattice, are defined by

$$\begin{aligned}
P &= \{\lambda \in \mathfrak{h}^* : \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z} \text{ for } i = 0, \dots, l\} \\
P_+ &= \{\lambda \in P : \langle \lambda, \alpha_i^\vee \rangle \geq 0 \text{ for } i = 0, \dots, l\} = \sum_{i=0}^{l} \mathbb{Z}_{>0}\Lambda_i + \mathbb{C}\delta, \\
P_{++} &= \{\lambda \in P_+ : \langle \lambda, \alpha_i^\vee \rangle > 0 \text{ for } i = 0, \dots, l\}, \\
\mathring{P} &= \sum_{i=1}^{l} \mathbb{Z}\overline{\Lambda}_i,
\end{aligned}$$

where $\Lambda_0, \dots, \Lambda_l$ denote the fundamental weights of $\mathfrak{g}(A)$ and $\overline{\Lambda}_1, \dots, \overline{\Lambda}_l$ denote the fundamental weights of $\mathring{\mathfrak{g}}$. Note the relationship $\Lambda_i = \overline{\Lambda}_i + a_i^\vee \Lambda_0$.

Representation theory of affine Kac–Moody algebras is a rich source of functions with modular properties. For example, the linear span of the normalized characters of highest weight $\mathfrak{g}(A)$-modules of fixed positive level is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$. This gives rise to the well-known modular $\mathcal{S}$-matrix, see [Kac90, Theorem 13.8].

For a general Kac–Moody algebra $\mathfrak{g}$, formal characters of $\mathfrak{g}$-modules V in category $\mathcal{O}$ record the dimensions of the weight spaces of V. One can then turn formal characters into holomorphic functions on $\mathfrak{h}$, and in the affine case develop their modularity properties referred to above with the use of theta functions.

For a highest weight $\mathfrak{g}(A)$-module $L(\Lambda)$, string functions are introduced to simplify the computation of its character. This is because the weights of an irreducible highest weight $\mathfrak{g}(A)$-module occur in *strings*. For each maximal weight, there is a corresponding string function that serves as a formal generating series for the weight spaces that belong to that string. These functions can then turned into holomorphic functions on $\mathbb{H}$. We are interested in string functions because they are modular forms whose congruence subgroup and weight is controlled entirely by data coming from $\mathfrak{g}(A)$ and level of the $\mathfrak{g}(A)$-module $L(\Lambda)$. We make use of string functions in Section in 4.10.

We first make some statements for a general Kac–Moody algebra $\mathfrak{g}$. In order for a character of a $\mathfrak{g}$-module V to make sense, the dimensions of the weight spaces must be finite, so we restrict our attention to the category $\mathcal{O}$.

Definition 4.5.1. Let V be a $\mathfrak{g}$-module in category $\mathcal{O}$. The formal character of V is a function $\mathrm{ch}_V : \mathfrak{h}^* \to \mathbb{Z}$ given by

$$\mathrm{ch}_V(\lambda) = \dim V_\lambda.$$

Let $\mathbb{Z}[P]$ denote the group algebra of P with $\mathbb{Z}$-basis given by the e^{λ} for $\lambda \in P$. The multiplication is given by

$$\mathrm{e}^{\lambda}\mathrm{e}^{\mu} = \mathrm{e}^{\lambda+\mu}.$$

Then we can regard formal character ch_V as a formal (possibly infinite) sum in $\mathbb{Z}[P]$ given by

$$\mathrm{ch}_V = \sum_{\lambda \in P(V)} \mathrm{mult}(\lambda)\mathrm{e}^{\lambda} \in \mathbb{Z}[P].$$

From (4.4.3) it follows that

$$\mathrm{e}^{-\Lambda}\mathrm{ch}_V \in \mathbb{C}[[\mathrm{e}^{-\alpha_1}, \dots, \mathrm{e}^{-\alpha_n}]],$$

or alternatively

$$\mathrm{e}^{-\Lambda}\mathrm{ch}_V \in \mathbb{C}[[\mathrm{e}^{-\delta}, \mathrm{e}^{-\alpha_1}, \dots, \mathrm{e}^{-\alpha_n}]].$$

We now restrict attention to integrable $\mathfrak{g}$-modules in category $\mathcal{O}$.

Definition 4.5.2. A $\mathfrak{g}$-module V in category $\mathcal{O}$ is integrable if e_i and f_i are locally nilpotent on V for $i = 1, \dots, n$.

Integrability ensures that the dimensions of the weight spaces are well behaved under the action of W. In other words, if V is an integrable $\mathfrak{g}$-module, then

$$\dim V_{\lambda} = \dim V_{w(\lambda)}$$

for all $w \in W$ and $\lambda \in \mathfrak{h}^*$. Furthermore, $P(V)$ is invariant under the action of W, see [Kac90, Proposition 3.7].

The $\mathfrak{g}$-module $L(\Lambda)$ is integrable if and only if $\Lambda \in P_+$. Much is known in this case about $\mathrm{ch}_{L(\Lambda)}$. Kac devised an analogue of the classical Weyl character formula in the setting of symmetrisable Kac–Moody algebras.

Theorem 4.5.3 (Weyl–Kac [Kac90, Theorem 10.4])**.** Let $\mathfrak{g}$ be a symmetrisable Kac–Moody algebra, and let $L(\Lambda)$ be an irreducible $\mathfrak{g}$-module of highest weight $\Lambda \in P_+$. Then

$$\mathrm{ch}_{L(\Lambda)} = \frac{\sum_{w \in W} \varepsilon(w)\mathrm{e}^{w(\Lambda+\rho)-\rho}}{\prod_{\alpha \in \Delta_+}(1-\mathrm{e}^{-\alpha})^{\mathrm{mult}(\alpha)}}. \tag{4.5.1}$$

Remark 4.5.4. Recall from (4.1.1) that $\mathfrak{g}(A)$ is symmetrisable when A is affine. In this case, characters of the $\mathfrak{g}(A)$-modules $L(\Lambda)$ for $\Lambda \in P_+$ may be computed using the Weyl–Kac character formula.

We now specialise to an affine algebra $\mathfrak{g}(A)$. An important integer attached to a $\mathfrak{g}(A)$-module is its *level*. The canonical central element c acts on $\mathfrak{g}(A)$ by the scalar operator $\langle \Lambda, c\rangle I_{L(\Lambda)}$. Moreover, $\langle \lambda, c\rangle = \langle \Lambda, c\rangle$ for all $\lambda \in P(\Lambda)$ by (4.4.3). We call $k := \langle \Lambda, c\rangle \in \mathbb{Z}_{>0}$ the level of $L(\Lambda)$.

When $\Lambda \in P_+$, $P(\Lambda)$ is highly structured in the affine case. All weights occur in so-called δ-strings, with each string in correspondence to a unique maximal weight. The weights also lie within a paraboloid on the weight lattice. These are consequences of Proposition 4.5.5.

Proposition 4.5.5 ([Kac90, Proposition 12.5])**.** Let $L(\Lambda)$ be an integrable highest weight module of positive level k over an affine Kac Moody-algebra. Then

(a) $P(\Lambda) = W \cdot \{\lambda \in P_+ : \lambda \leq \Lambda\}$,

(b) $P(\Lambda) = (\Lambda + Q) \cap$ convex hull of $W \cdot \Lambda$,

(c) $P(\Lambda)$ lies in the paraboloid

$$\{\lambda \in \mathfrak{h}^*_{\mathbb{R}} : |\overline{\lambda}| + 2k(\lambda|\Lambda_0) \leq |\Lambda|^2, \langle \lambda, c\rangle = k\}.$$

The intersection of $P(\Lambda)$ with the boundary of this paraboloid is $W \cdot \Lambda$

(d) For $\lambda \in P(\Lambda)$, the set of $t \in \mathbb{Z}$ for which $\lambda - t\delta \in P(\Lambda)$, is an interval $[-p, \infty)$ where $p \geq 0$.

Definition 4.5.6. A weight $\lambda \in P(\Lambda)$ is called maximal if $\lambda + \delta \notin P(\Lambda)$. Let $\max(\Lambda)$ denote the set of maximal weights in $P(\Lambda)$.

Since $P(\Lambda)$ is invariant and $w\delta = \delta$ for all $w \in W$, it is clear that $\max(\Lambda)$ is W-invariant. If $\Lambda \in P_+$, then condition (a) of Proposition 4.5.5 says that any $\lambda \in P(\Lambda)$ is W-equivalent to a unique dominant weight. It follows from condition (d) of Proposition 4.5.5 that for every $\mu \in P(\Lambda)$ there exists a unique $\lambda \in \max(\Lambda)$ such that $\mu = \lambda - n\delta$ for some integer $n \geq 0$. Thus each $\lambda \in P(\Lambda)$ is W-equivalent to a unique dominant maximal weight.

Remark 4.5.7. See Figure 4.5.10 to see the weights of the $A_1^{(1)}$-module $L(\Lambda_0)$.

The following two results assist in the computation of characters.

Proposition 4.5.8 ([Kac90, Proposition 12.6])**.** The map $\lambda \mapsto \overline{\lambda}$ defines a bijection from $\max(\lambda) \cap P_+$ onto $kC_{\text{af}} \cap (\overline{\Lambda} + \overline{Q})$. Moreover, the set of dominant maximal weights is finite.

Lemma 4.5.9 ([Kac90, Lemma 12.6])**.** Let $A = X_l^{(r)}$ where $X = \text{A}, \text{D}$ or E. Let $\Lambda \in P_+$ have level 1. Then

$$\max(\Lambda) = W \cdot \Lambda = T \cdot \Lambda = T \cdot \Lambda.$$

Example 4.5.10 ([Car05, Section 20.4])**.** We wish to compute the characters of the $A_1^{(1)}$-modules $L(\Lambda_0)$ and $L(\Lambda_1)$. We have the GCM

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

By symmetry it is sufficient to consider $L(\Lambda_0)$. In this case,

$$\begin{aligned} \delta &= \alpha_0 + \alpha_1, & \theta &= \alpha_1, \\ Q &= \mathbb{Z}\alpha_0 + \mathbb{Z}\alpha_1, & \mathring{Q} &= \mathbb{Z}\alpha_1, \\ W &= \langle r_0, r_1\rangle, & \mathring{W} &= \{1, r_1\}. \end{aligned}$$

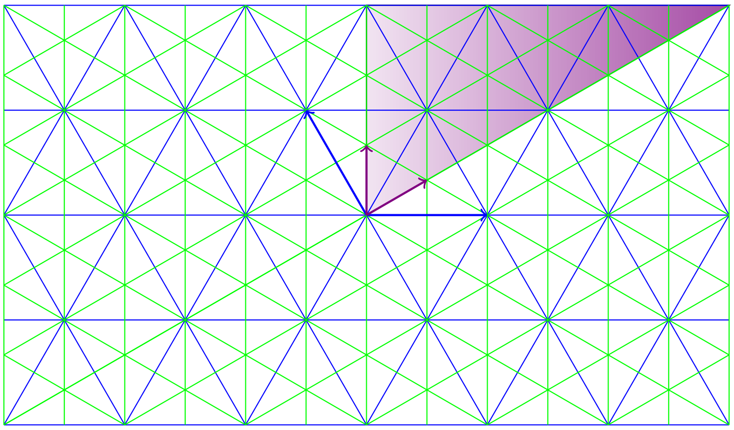

Figure 4.2: The root lattice, weight lattice and cone of dominant weights for $A_1^{(1)}$.

In addition,

$$C_{\text{af}} = \{\lambda \in \mathring{\mathfrak{h}}_{\mathbb{R}}^* : 0 \leq \langle\lambda|\alpha_1\rangle \leq 1\} \quad \text{and} \quad \mathring{\mathfrak{h}}^* = \mathbb{C}\alpha_1.$$

Now,

$$\left(\overline{\Lambda}_0 + \mathring{Q}\right) \cap C_{\text{af}} = \{m\alpha_1 : m \in \mathbb{Z}, 0 \leq 2m \leq 1\} = \{0\}.$$

Thus $L(\Lambda_0)$ has a unique dominant maximal weight by Proposition 4.5.8. This must be the highest weight Λ_0. Applying Lemma 4.5.9,

$$\max(\Lambda_0) = W \cdot \Lambda_0.$$

The action of W on the basis $\Lambda_0, \alpha_1, \delta$ of $\mathfrak{h}^*$ is given by

$$r_0(\Lambda_0) = \Lambda_0 + \alpha_1 - \delta, \qquad r_0(\alpha_1) = -\alpha_1 + 2\delta, \qquad r_0(\delta) = \delta \tag{4.5.2}$$

$$r_1(\Lambda_0) = \Lambda_0, \qquad r_1(\alpha_1) = -\alpha_1 \qquad r_1(\delta) = \delta. \tag{4.5.3}$$

Using (4.3.2) we compute the following family of translations for $m \in \mathbb{Z}$

$$\begin{aligned} t_{m\alpha_1}(\Lambda_0) &= \Lambda_0 + m\alpha_1 - m^2\delta, \\ t_{m\alpha_1}(\alpha_1) &= \alpha_1 - 2m\delta, \\ t_{m\alpha_1}(\delta) &= \delta. \end{aligned}$$

Thus

$$\max(\Lambda_0) = \left\{\Lambda_0 + m\alpha_1 - m^2\delta : m \in \mathbb{Z}\right\}.$$

We deduce

$$P(\Lambda_0) = \left\{\Lambda_0 + m\alpha_1 - m^2\delta - k\delta : m \in \mathbb{Z}, k \in \mathbb{Z}_{\geq 0}\right\}.$$

We now compute $\operatorname{ch}_{L(\Lambda_0)}$ using the Weyl–Kac character formula. A good approach is to compute the numerator and denominator of (4.5.1) separately. Using Proposition 4.3.5 we split the numerator of (4.5.1) into a double sum,

$$\begin{aligned} \sum_{w \in W} \varepsilon(w) \mathrm{e}^{w(\Lambda_0+\rho)-\rho} &= \sum_{\mathring{w} \in \mathring{W}} \sum_{\mu \in \mathbb{Z}\alpha_1} \varepsilon(\mathring{w}) \mathrm{e}^{\mathring{w} t_\mu(\Lambda_0+\rho)-\rho} \\ &= \sum_{\mathring{w} \in \mathring{W}} \varepsilon(\mathring{w}) \sum_{n \in \mathbb{Z}} \mathrm{e}^{\mathring{w} t_{n\alpha_1}(\Lambda_0+\rho)-\rho}. \end{aligned} \tag{4.5.4}$$

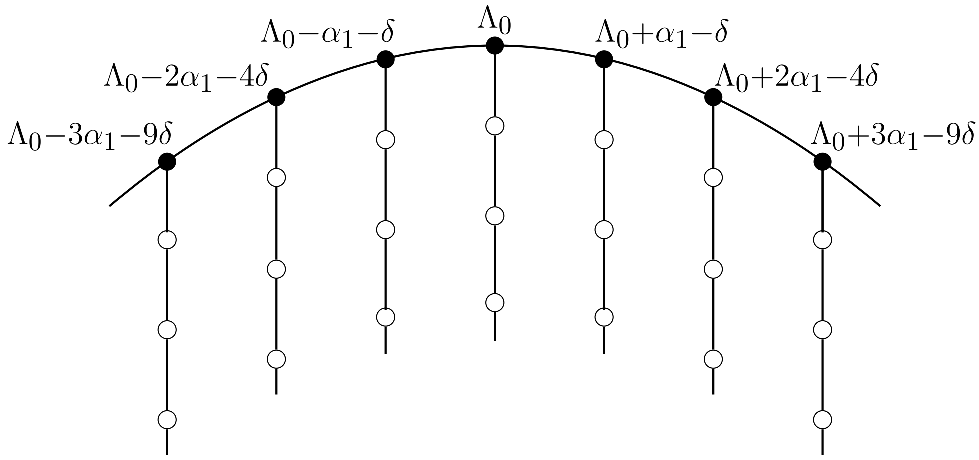

Figure 4.3: The δ-strings of weights for the fundamental module $L(\Lambda_0)$ of $A_1^{(1)}$. Weights λ of $L(\Lambda_0)$ lie in the parabola $|\overline{\lambda}|^2 + 2(\lambda|\Lambda_0) \leq 0$ with $\langle \lambda, c\rangle = 1$ by condition (c) of Proposition 4.5.5.

Consider the following computations,

$$\begin{aligned} t_{n\alpha_1}(\Lambda_0+\rho) &= 3\Lambda_0 + \left(3n+\tfrac{1}{2}\right)\alpha_1 - (3n^2+n)\delta \\ &= \Lambda_0 + 3n\alpha_1 - (3n^2+n)\delta \\ s_1 t_{n\alpha_1} &= 3\Lambda_0 - \left(3n+\tfrac{1}{2}\right)\alpha_1 - (3n^2+n)\delta \\ &= \Lambda_0 - (3n+1)\alpha_1 - (3n^2+n)\delta. \end{aligned}$$

Equation (4.5.4) becomes

$$\sum_{w\in W} \varepsilon(w) \mathrm{e}^{w(\Lambda_0+\rho)-\rho} = \mathrm{e}^{\Lambda_0} \sum_{n\in\mathbb{Z}} \left(\mathrm{e}^{3n\alpha_1} - \mathrm{e}^{(3n+1)\alpha_1}\right) \mathrm{e}^{-n(3n+1)\delta}$$

Writing $z := \mathrm{e}^{-\alpha_1}$ and $\mathrm{e}^{-\delta} = q$ yields

$$\mathrm{e}^{\Lambda_0} \sum_{n\in\mathbb{Z}} \left(z^{-3n} - z^{3n+1}\right) q^{n(3n+1)}. \tag{4.5.5}$$

Factorising (4.5.5) using Watson's quintuple identity [GR04, Exercise 5.6], and then applying the Jacobi triple product identity [GR04, Section 1.6] yields

$$\mathrm{e}^{\Lambda_0}(z, q/z; q)_\infty (q^2, -qz, -q/z; q^2)_\infty = \mathrm{e}^{\Lambda_0}(z, q/z; q)_\infty \sum_{n\in\mathbb{Z}} z^{-n} q^{n^2}.$$

The denominator of (4.5.1) with the use of Proposition 4.2.5 is written as

$$(q, z, q/z; q)_\infty.$$

Recalling that $(q;q)_\infty^{-1}$ is the generating function for partitions we see

$$\begin{aligned} \mathrm{ch}_{L(\Lambda_0)} &= \frac{\mathrm{e}^{\Lambda_0} \sum_{n\in\mathbb{Z}} z^{-n} q^{n^2}}{(q;q)_\infty} \\ &= \sum_{n\in\mathbb{Z}} \sum_{k\geq 0} p(k) \mathrm{e}^{\Lambda_0 + n\alpha_1 - n^2\delta - k\delta}. \end{aligned}$$

Since all of the weights of the integrable module $L(\Lambda)$ come in δ-stings, the computation of its character can be reduced to computing the so called string functions.

Definition 4.5.11. Suppose $\Lambda \in P_+$ and $\lambda \in \max(\Lambda)$. The string function through λ, denoted b^Λ_λ, is the generating series

$$b^\Lambda_\lambda := \sum_{n=0}^{\infty} \mathrm{mult}_{L(\lambda)}(\lambda - n\delta) q^n,$$

where $q := \mathrm{e}^{-\delta}$.

The string functions are invariant under the W-action on $P(\Lambda)$. In particular, $b^\Lambda_{w(\lambda)} = b^\Lambda_\lambda$ for all $w \in W$. Since every $\lambda \in P(\Lambda)$ belongs to a unique δ-string, $\mathrm{stab}(\lambda) \cap T = \{1\}$ and the symmetry above, we have the equality

$$\mathrm{ch}_{L(\Lambda)} = \sum_{\lambda \in \max(\Lambda)} b^\Lambda_\lambda \mathrm{e}^\lambda = \sum_{\substack{\lambda \in \max(\Lambda) \\ \lambda \mod T}} b^\Lambda_\lambda \left(\sum_{t \in T} \mathrm{e}^{t(\lambda)} \right). \tag{4.5.6}$$

This naturally leads to classical theta functions.

Definition 4.5.12. For $\lambda \in \mathfrak{h}^*$ of level $\langle \lambda, c \rangle = k > 0$ define

$$\Theta_\lambda := \mathrm{e}^{-\frac{|\lambda|^2}{2k}\delta} \sum_{t \in T} \mathrm{e}^{t(\lambda)} \quad .$$

For $\Lambda \in P^k_+$ and $\lambda \in P(\Lambda)$ introduce the numbers

$$m_\Lambda := \frac{|\Lambda + \rho|^2}{2(k + h^\vee)} - \frac{|\rho|^2}{2h^\vee} \quad \text{and} \quad m_{\Lambda,\lambda} := m_\Lambda - \frac{|\lambda|^2}{2k}.$$

These are used to normalize the characters and string functions respectively so they have modular properties, but this is not our concern at present (as we have not introduced these functions on the upper half-plane yet). The normalizations are given by

$$\chi_\Lambda := \mathrm{e}^{-m_\Lambda \delta} \mathrm{ch}_{L(\Lambda)}$$
$$c^\Lambda_\lambda := \mathrm{e}^{-m_{\Lambda,\lambda}\delta} b^\Lambda_\lambda.$$

The normalized string function c^Λ_λ has some useful symmetries. Again we have $c^\Lambda_{w(\lambda)} = c^\Lambda_\lambda$ for all $w \in W$. Furthermore,

$$c^\Lambda_{w(\lambda)+k\gamma+a\delta} = c^\Lambda_\lambda \quad \text{for all} \quad \lambda \in \mathring{W}, \gamma \in M, a \in \mathbb{C}.$$

It follows that there are finitely many distinct string functions for a given highest weight. Combining (4.5.6) and the definitions of Θ_λ and c^Λ_λ, we can write

$$\begin{aligned} \chi_\Lambda &= \sum_{\substack{\lambda \in P^k \mod \mathbb{C}\delta \\ \lambda \mod T}} c^\Lambda_\lambda \Theta_\lambda \\ &= \sum_{\lambda \in P^k \mod kM + \mathbb{C}\delta} c^\Lambda_\lambda \Theta_\lambda. \end{aligned} \tag{4.5.7}$$

Example 4.5.13 ([Wak01, Example 2.1.5])**.** Consider the problem of computing the string functions for the $A_1^{(1)}$-module $L(2\Lambda_0)$.

Noting for each $m \in \mathbb{Z}$ we have

$$t_{m\alpha_1}(2\Lambda_0) = 2\Lambda_0 + 2m\alpha_1 - 2m^2\delta,$$

the Weyl-orbit of Λ is

$$W \cdot \Lambda = \{2\Lambda_0 + 2m\alpha_1 - 2m^2\delta : m \in \mathbb{Z}\}.$$

Considering the convex hull of $W \cdot \Lambda$ and applying Proposition 4.5.5b we see the weights are given by

$$P(\Lambda) = \left\{\Lambda + p\alpha_1 + q\delta : p, q \in \mathbb{Z}, q \leq -\frac{1}{2}p^2\right\}.$$

Thus the maximal weights are

$$\max(2\Lambda_0) = \left\{2\Lambda_0 + m\alpha_1 - \frac{m^2}{2}\delta : m \in 2\mathbb{Z}\right\} \cup \left\{2\Lambda_0 + m\alpha_1 - \left(\frac{m^2+1}{2}\right)\delta : m \in 1 + 2\mathbb{Z}\right\}.$$

Hence $\max(2\Lambda_0)$ is the image of two Weyl-orbits,

$$\begin{aligned}\max(2\Lambda_0) &= W \cdot (2\Lambda_0) \cup W \cdot (2\Lambda_0 + \alpha_1 - \delta)\\ &= W \cdot (2\Lambda_0) \cup W \cdot (2\Lambda_0 - \alpha_0).\end{aligned}$$

There are two string functions $b_{2\Lambda_0}^{2\Lambda_0}$ and $b_{2\Lambda_0-\alpha_0}^{2\Lambda_0}$ to compute. We first compute the following translations

$$t_{m\alpha_1}(2\Lambda_0 + \rho) = 2\Lambda_0 + \rho + 4m\alpha_1 - (4m^2 + m)\delta, \tag{4.5.8a}$$

$$t_{m\alpha_1}(2\Lambda_1 + \rho) = 2\Lambda_0 + \rho + (4m+1)\alpha_1 - \frac{1}{2}\binom{4m+2}{2}\delta + \frac{1}{2}\delta, \tag{4.5.8b}$$

$$r_1 t_{m\alpha_1}(2\Lambda_0 + \rho) = 2\Lambda_0 + \rho - (4m+1)\alpha_1 - \frac{1}{2}\binom{4m+1}{2}\delta, \tag{4.5.8c}$$

$$r_1 t_{m\alpha_1}(2\Lambda_1 + \rho) = 2\Lambda_0 + \rho - \frac{1}{2}\binom{4m+3}{2}\delta + \frac{1}{2}\delta. \tag{4.5.8d}$$

Substituting the equations from (4.5.8) into the numerator of the Weyl–Kac character formula, and then applying Jacobi triple product identity twice we have

$$\begin{aligned}e^{-2\Lambda_0}\left(\mathrm{ch}_{L(2\Lambda_0)} - q^{1/2}\mathrm{ch}_{L(2\Lambda_1)}\right) &= \frac{(q^{1/2}, z, q^{1/2}/z; q^{1/2})_\infty}{(q, z, q/z; q)_\infty}\\ &= (q^{1/2}, zq^{1/2}, q^{1/2}/z; q)_\infty\\ &= \frac{(q^{1/2}, q)_\infty}{(q;q)_\infty}(q, zq^{1/2}, q^{1/2}/z; q)_\infty\\ &= \frac{(q^{1/2}; q)_\infty}{(q;q)_\infty} \cdot \sum_{j\in\mathbb{Z}}(-1)^j z^j q^{j^2/2}.\end{aligned} \tag{4.5.9}$$

The sum over j is alternating in (4.5.9). When j is even, this corresponds to a contribution of $b^{2\Lambda_0}_{2\Lambda_0}$ from $\mathrm{ch}_{L(2\Lambda_0)}$ to $(q^{1/2};q)_\infty/(q;q)_\infty$ and no contribution from $q^{1/2}\mathrm{ch}_{L(2\Lambda_1)}$. When j is odd, this is a contribution from $q^{1/2}b^{2\Lambda_1}_{2\Lambda_1-\alpha_1}$. Noticing the symmetry $b^{2\Lambda_0}_{2\Lambda_0-\alpha_0} = b^{2\Lambda_1}_{2\Lambda_1-\alpha_1}$ we obtain

$$b^{2\Lambda_0}_{2\Lambda_0} - q^{1/2}\cdot b^{2\Lambda_0}_{2\Lambda_0-\alpha_0} = \frac{(q^{1/2};q)_\infty}{(q;q)_\infty}. \tag{4.5.10}$$

Letting $q^{1/2} \mapsto -q^{1/2}$ yields

$$b^{2\Lambda_0}_{2\Lambda_0} + q^{1/2}\cdot b^{2\Lambda_0}_{2\Lambda_0-\alpha_0} = \frac{(-q^{1/2};q)_\infty}{(q;q)_\infty}. \tag{4.5.11}$$

Solving (4.5.10) and (4.5.11) yields

$$b^{2\Lambda_0}_{2\Lambda_0} = \frac{1}{2}\cdot\frac{(q^{1/2};q)_\infty + (-q^{1/2};q)_\infty}{(q;q)_\infty}$$

$$b^{2\Lambda_0}_{2\Lambda_0-\alpha_0} = \frac{1}{2}\cdot\frac{(-q^{1/2};q)_\infty - (q^{1/2};q)_\infty}{(q;q)_\infty}.$$

4.6 String functions on $\mathbb{H}$ and modularity

We outline how holomorphic functions on the complex upper half plane are built from formal characters. We describe their regions of absolute convergence and give details on a modularity result concerning the string functions that we use in Section 4.10.

We can regard $\mathrm{e}^\lambda \in \mathbb{Z}[P]$ in the previous section as functions on $\mathfrak{h}$ or $\mathfrak{h}^*$ to $\mathbb{C}$. For any $h \in \mathfrak{h}$ or $\mu \in \mathfrak{h}^*$, define $\mathrm{e}^\lambda(h) := \mathrm{e}^{\langle\lambda,h\rangle}$ and $\mathrm{e}^\lambda(\mu) := \mathrm{e}^{\langle\lambda|\mu\rangle}$ respectively. Now that we are starting to consider characters as $\mathbb{C}$-valued functions, we need to consider issues of convergence.

Definition 4.6.1. Let $Y(L(\Lambda)) \subset \mathfrak{h}$ be the set where $\mathrm{ch}_{L(\Lambda)}$ converges absolutely. For $h \in Y(L(\Lambda))$ we may write

$$h \mapsto \mathrm{ch}_{L(\Lambda)}(h) = \sum_{\lambda\in P(\Lambda)} \mathrm{mult}(\lambda)e^{\langle\lambda,h\rangle}.$$

From the easy estimate

$$|c^\Lambda_\lambda(h)| \le |\chi_\Lambda(h)| = |\mathrm{e}^{-m_\Lambda\langle\delta,h\rangle}|\cdot|\mathrm{ch}_{L(\Lambda)}(h)|,$$

both c^Λ_λ and χ_Λ both absolutely converge on $Y(L(\Lambda))$. Define the following subset of $\mathfrak{h}$

$$Y := \{h\in\mathfrak{h} : \sum_{\alpha\in\Delta^+}(\mathrm{mult}(\alpha))|\mathrm{e}^{-\langle\alpha,h\rangle}| < \infty\}.$$

Geometrically speaking, $Y(L(\Lambda))$ is convex since $|\mathrm{e}^\lambda| : \mathfrak{h}\to\mathbb{C}$ is convex function. It is important to note that $\mathrm{ch}_{L(\Lambda)}$, χ_Λ and c^Λ_λ are holomorphic functions on $Y(L(\Lambda))$.

Proposition 4.6.2 ([Kac90, Proposition 11.10])**.** Let A be an indecomposable symmetrisable Cartan matrix and let $L(\Lambda)$ be an irreducible $\mathfrak{g}(A)$-module with highest weight $\Lambda \in P_+$, such that $\langle \Lambda, \alpha_i \rangle \neq 0$ for some i. Then $Y(L(\Lambda)) = Y$ and $\mathrm{ch}_{L(\Lambda)}$ is a holomorphic function on $Y(L(\Lambda))$.

Applying Proposition 4.2.5 and Corollary 4.2.7 we obtain the second equality in

$$Y(L(\Lambda)) = Y = \{h \in \mathfrak{h} : \mathrm{Re}\langle \delta, h \rangle > 0\}\,, \tag{4.6.1}$$

the latter set being a much simpler description. We now are able to view the normalized string functions c_λ^Λ as functions on $\mathbb{H}$. Consider the identification

$$\mathfrak{h} \cong \mathfrak{h}^* = \mathring{\mathfrak{h}}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta,$$

and write $h \in \mathfrak{h}$ as

$$h = 2\pi i(z - \tau\Lambda_0 + t\delta) \quad \text{for some} \quad z \in \mathring{\mathfrak{h}}^*, \tau, t \in \mathbb{C}.$$

Then by (4.6.1),

$$Y(L(\Lambda)) = \{(\tau, z, t) : z \in \mathring{\mathfrak{h}}^*, \tau, t \in \mathbb{C} \text{ with } \mathrm{Im}\,\tau > 0\},$$

and we write

$$c_\lambda^\Lambda(\tau) := q^{-m_{\Lambda,\lambda}} \sum_{n=0}^{\infty} \mathrm{mult}(\lambda - n\delta) q^n \quad \text{for} \quad q = \mathrm{e}^{2\pi i\tau}, \tau \in \mathbb{H}.$$

After a long story that involves studying the modular transformation properties of $\Theta_\Lambda(h) = \Theta_\Lambda(\tau, z, t)$ (see Proposition 13.7 in [Kac90]) and the modular $\mathcal{S}$-matrix, one can prove that the $c_\lambda^\Lambda(\tau)$ are modular forms whose congruence subgroup and weight depend only data coming from the affine algebra $\mathfrak{g}(A)$ and the level of the module.

Theorem 4.6.3 ([Kac90, Theorem 13.12])**.** Let $\mathfrak{g}(A)$ be an affine algebra of type $X_N^{(r)}$ and rank $l+1$, and let $\Lambda \in P_+^k$ with $k > 0$. Let $s = \mathrm{lcm}\{k, h^\vee, k + h^\vee, N\}$. Then the string functions $c_\lambda^\Lambda(\tau)$ are modular forms of weight $-\frac{1}{2}l$ with multiplier system $(\pm 1)^l$, with respect to $\Gamma(s)$.

4.7 Virasoro algebra

We make a brief note on the Virasoro algebra, its highest weight representations and minimal models. We follow the short summary given in [Lee12] The characters arising from minimal models in a special case will be related to the modularity of $F_{A,B,C}(q)$ when $A = C_{\mathrm{A}_1} \otimes C_{T_{k-1}}^{-1}$, considered in Section 4.9.

Definition 4.7.1. The Virasoro algebra Vir is a complex Lie algebra spanned by the elements L_n, $n \in \mathbb{Z}$ and the central element c satisfying the commutation relations

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}.$$

Similar to Definition 4.4.4, we call $\rho : \mathrm{Vir} \to \mathrm{End}(V)$ a highest weight representation with highest weight $(c, h) \in \mathbb{C}^2$ if there exists a non-zero $v_0 \in V$ such that the following conditions are satisfied

(a) $\rho(L_0)v_0 = hv_0$,

(b) $\rho(c)v_0 = cv_0$,

(c) $\rho(L_n)v_0 = 0 > 0$,

(d) V is spanned by elements of the form $\rho(L_{-n_1})\rho(L_{-n_2})\cdots\rho(L_{-n_k})v_0$ with $n_1 \geq n_2 \geq \cdots \geq n_k > 0$.

Remark 4.7.2. Note that on the left hand side of $\rho(c)v_0 = cv_0$, c denotes the central element in the algebra and on the right it denotes the central charge, the scalar occurring in $(c, h) \in \mathbb{C}^2$. This abuse of notation is standard.

Each highest weight representation of the Virasoro algebra is parametrised by $(c, h) \in \mathbb{C}^2$. We have the direct sum decomposition

$$V = \bigoplus_{n\geq 0} V_n,$$

where $V_0 = \mathbb{C}v_0$ and V_n is spanned by $\rho(L_{-n_1})\rho(L_{-n_2})\cdots\rho(L_{-n_k})v_0$ subject to the condition $n_1 + \cdots + n_k = n$ and $n_i \in \mathbb{Z}$. The character of the highest weight module V is defined by

$$\mathrm{tr}_V q^{L_0 - c/24} = \sum_{n=0}^{\infty} (\dim V_n) q^{h-c/24+n}.$$

For each $(c, h) \in \mathbb{C}^2$ there is a unique irreducible module of the Virasoro algebra. For a pair of relatively prime positive integers (p, p') with $2 \leq p < p'$, the minimal model $M(p, p')$ consists of the irreducible highest weight Virasoro modules parameterised by $(c, h_{r,s})$, where $1 \leq r \leq p-1$, $1 \leq s \leq p'-1$,

$$c = 1 - 6\frac{(p'-p)^2}{p'p} \quad \text{and} \quad h_{r,s} = \frac{(p'r - ps)^2 - (p'-p)^2}{4pp'}.$$

The salient point is that the character $\chi_{r,s}^{(p,p')}$ of the unique representation corresponding to $(c, h_{r,s})$ is given by the Rocha-Caridi formula [FF84, RC85]. In particular, for $1 \leq r \leq p-1$ and $1 \leq s \leq p'-1$ we have

$$\chi_{r,s}^{(p,p')} = \frac{q^{h_{r,s}^{(p,p')} - c/24}}{(q)_\infty} \sum_{n=-\infty}^{\infty} \left(q^{pp'n^2 + (rp' - sp)n} - q^{(pn+r)(p'n+s)} \right), \tag{4.7.1}$$

together with the useful symmetry

$$\chi_{r,s}^{(p,p')} = \chi_{p'-r,p-s}^{(p,p')}. \tag{4.7.2}$$

In particular, the $M(2, 2k+1)$ minimal model has a family of characters naturally arise in the study of Nahm's conjecture in Section 4.9.

4.8 A Theorem of Lee

Using Y-systems and dilogarithms, Lee in [Lee13] proved that the solutions to $Q = (1-Q)^A$ are all torsion, providing strong evidence that there is at least one modular triple for any choice of $A = C_X \otimes C_{X'}^{-1}$. Suppose I is an index set and we write any solution to $Q = (1-Q)^A$ as $Q = (Q_i)_{i \in I}$.

Theorem 4.8.1 ([Lee13, Theorem 1.3])**.** Let $A = C_X \otimes C_{X'}^{-1}$ where (X, X') is a pair of Dynkin diagrams of ADE or T type. For every solution $Q = (Q_i)_{i\in I}$ of $Q = (1-Q)^A$, $\xi_Q = \sum_{i\in I}[Q_i] \in \mathcal{B}(F)$ is a torsion element where $F = \mathbb{Q}((Q)_{i\in I})$.

Interestingly, the following dilogarithm identity links the unique solution in $(0,1)^{|I|}$ of $Q = (1-Q)^A$ to the effective central charge.

Theorem 4.8.2 ([Lee12, Theorem 4.2.5])**.** The unique solution $Q = (Q_\mathbf{i}) \in (0,1)^{|I|}$ with for all $i \in I$ to $Q = (1-Q)^A$ satisfies the following dilogarithm identity

$$\sum_{i\in I} L(x_i) = \frac{h_X r_X r_{X'}}{h_X + h_{X'}} L(1).$$

4.9 Andrews–Gordon experiments

In this section we consider the family of matrices $A = C_{\mathrm{A}_1} \otimes C_{\mathrm{T}_{k-1}}^{-1}$. These occur in the famous Andrews–Gordon identity, which generalise the Rogers–Ramanujan identity to arbitrary level.

Theorem 4.9.1 (Andrews–Gordon)**.** For two integers k and l such that $k \geq 2$ and $1 \leq l \leq k$,

$$\sum_{n_1,\dots,n_{k-1}\geq 0} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_l+\cdots+N_{k-1}}}{(q)_{n_1}\cdots(q)_{n_{k-1}}} = \prod_{r\not\equiv 0,\pm i \pmod{2k+1}} \frac{1}{1-q^r}, \tag{4.9.1}$$

where $N_j = n_j + \cdots + n_{k-1}$ for $j \leq k-1$ and $N_k := 0$.

Expanding the quadratic from in the exponent of q in (4.9.1) for each $1 \leq i \leq k$ yields

$$N_1^2 + \cdots + N_{k-1}^2 + N_l + \cdots + N_{k-1} = \sum_{r,s=1}^{k-1} \min\{r,s\} n_r n_s + \sum_{j=l}^{k-1} (k-i) n_j.$$

Thus we see the occurence of

$$A = C_{\mathrm{A}_1} \otimes C_{T_{k-1}}^{-1} = 2 \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & k-1 \end{pmatrix},$$

and the family of B-vectors B_i, defined by

$$B_i = \frac{1}{2}A(e_{k-1} - e_i) \quad \text{for} \quad i = 1, 2, \ldots, k-1 \quad \text{and} \quad B_k = \frac{1}{2}Ae_{k-1}.$$

The right hand side of (4.9.1) is related to the characters of the minimal model $M(2, 2k+1)$ given by the Rocha-Caridi formula in (4.7.1). This is seen by employing the Jacobi triple product identity to re-write the right hand side of (4.7.1) in the form occurring in the right hand side of (4.9.1) up to a rational power of q. In particular, we are considering the family of characters

$$\chi_{1,s}^{(2,2k+1)}(q) = \frac{q^{h_{1,s}^{(2,2k+1)} - c/24}}{(q)_\infty} \sum_{n=-\infty}^{\infty} \left(q^{2(2k+1)n^2 + (2k+1-2s)n} - q^{(2n+1)\left((2k+1)n+s\right)}\right) \quad \text{for} \quad 1 \le s < 2k+1,$$

where

$$c = 1 - 3\frac{(2k+1-2s)^2}{2k+1} \quad \text{and} \quad h_{1,s} = \frac{(2k+1-2s)^2 - (2k-1)^2}{8(2k+1)}.$$

Since $\chi_{1,s}(q) = \chi_{1,2k+1-s}(q)$ by (4.7.2), there are exactly k characters, each modular. Thus $q^{h_{1,k+1-i} - c/24} \times$ (4.9.1) matches with $\chi_{1,k+1-i}^{(2,2k+1)}(q)$. Thus each of the B_i are modular B-vectors for A. Are these the only possible modular B-vectors? In other words, we are asking whether the only modular B-vectors for A the ones that correspond to characters arising from the $M(2, 2k+1)$ minimal model. For small ranks, Keegan and Nahm in [KN11] investigate this question with computational experiments. For computational reasons they restrict their searches to the domain $-8 \le b_i \le 8$. All of their searches return only the family B_i of modular B-vectors.

We describe an alternative general approach that does not require restrictions on the b_i for computations involving small rank examples. Recall from Section 3.1 that modular (A, B, C) must satisfy the infinite number of polynomial equations

$$\left(c_p - \frac{1}{p!}c_1^p\right)(B, \zeta, \tilde{A}^{-1}) = 0 \quad \text{for} \quad p = 2, 3, \ldots,$$

where the $c_p \in \mathbb{Q}[B, \zeta, \tilde{A}^{-1}]$ are defined in [VZ11], and originate from the asymptotics of $F_{A,B,C}(q)$. Once A is fixed, so is ζ, and we obtain polynomials in $\mathbb{Q}(Q_1^0, \ldots, Q_r^0)[b_1, \ldots, b_r]$ of which the entries of modular B must be a root. We use `Magma` to compute these polynomials when $A = C_{\mathrm{A}_1} \otimes C_{\mathrm{T}_2}^{-1}$ and $A = C_{\mathrm{A}_1} \otimes C_{\mathrm{T}_3}^{-1}$. `Magma` is then used confirm the only B vectors are the ones above.

Example 4.9.2. Consider when $A = C_{\mathrm{A}_1} \otimes C_{\mathrm{T}_2}^{-1}$. All solutions lie in the totally real number field $\mathbb{Q}(\alpha)$ where the minimal polynomial of α is $x^3 + 2x^2 - x - 1$. Using the asymptotic method described above, we computed the following polynomials in $\mathbb{Q}(\alpha)[b_1, b_2]$,

$$c_2 - \frac{1}{2}c_1^2 = \frac{1}{294}(-2577\alpha^2 - 3720\alpha + 4634)b_1^3 + \frac{1}{49}(687\alpha^2 + 991\alpha - 1236)b_1^2 b_2$$
$$+ \cdots + \frac{1}{588}(1040\alpha^2 + 1503\alpha - 1866)b_2,$$

$$c_3 - \frac{1}{6}c_1^3 = \frac{1}{4116}(-179878\alpha^2 - 259904\alpha + 324095)b_1^5 + \frac{1}{2058}(242633\alpha^2 + 350568\alpha - 437166)b_1^4 b_2 + \cdots + \frac{1}{49392}(612981\alpha^2 + 886032\alpha - 1104094)b_2.$$

Using Magma, the only $(b_1, b_2) \in \mathbb{Q}^2$ that are roots of both polynomials are $(0,0), (0,1)$ and $(1,2)$.

Example 4.9.3. Consider when $A = C_{\mathrm{A}_1} \otimes C_{\mathrm{T}_3}^{-1}$. All solutions lie in the totally real number field $\mathbb{Q}(\alpha)$ where the minimal polynomial of α is $x^3 - x^2 - 2x + 1$. We computed the following polynomials in $\mathbb{Q}(\alpha)[b_1, b_2, b_3]$,

$$c_2 - \frac{1}{2}c_1^2 = \frac{1}{486}(-16537\alpha^2 - 27329\alpha + 36810)b_1^3 + \frac{1}{243}(14152\alpha^2 + 23388\alpha - 31509)b_1^2 b_2 + \cdots + \frac{1}{972}(69\alpha^2 + 116\alpha - 141)b_3,$$

$$c_3 - \frac{1}{2}c_1^3 = \frac{1}{8748}(-3076243\alpha^2 - 5084107\alpha + 6849779)b_1^5 + \frac{1}{4374}(4594124\alpha^2 + 7592717\alpha - 10229635)b_1^4 b_2 + \cdots + \frac{1}{209952}(24045\alpha^2 + 41452\alpha - 49275)b_3,$$

$$c_4 - \frac{1}{2}c_1^4 = \frac{1}{944784}(-1717314947\alpha^2 - 2838212520\alpha + 3823913286)b_1^7 + \cdots + \frac{1}{50388480}(13408589\alpha^2 + 22587752\alpha - 28773033)b_3.$$

Using Magma, the only vectors $(b_1, b_2, b_3) \in \mathbb{Q}^3$ that are roots of both polynomials are

$$(0,0,0), (0,0,1), (0,1,2) \quad \text{and} \quad (1,2,3).$$

4.10 Modularity in the $A = C_{\mathbf{A}_{n-1}} \otimes C_{\mathbf{A}_{k-1}}^{-1}$ case

In this section we prove Proposition 4.0.1. Recall that we wish to consider string functions of the unique irreducible $A_{n-1}^{(1)}$-module of highest weight $\Lambda = k_0\Lambda_0 + k_j\Lambda_j$, $j \in \{1, \ldots, n-1\}$. To state Georgiev's formula more compactly we define the rational constant

$$T(n,k,k_0,j) := \frac{\langle \Lambda + \rho | \Lambda + \rho \rangle}{2(k + h^\vee)} - \frac{\langle \rho | \rho \rangle}{2h^\vee} - \frac{\langle \Lambda | \Lambda \rangle}{2k}.$$

Theorem 4.10.1. [Geo94] Let Λ be as above and $\lambda \in \mathring{P}$. The normalised string functions c_λ^Λ of the $\mathrm{A}_n^{(1)}$-module $L(\Lambda)$ are

$$(4.10.1) \qquad c_\lambda^\Lambda = \frac{q^{T(n,k,k_0,j)}}{(q)_\infty^n} \sum_\bullet \frac{q^{\frac{1}{2}\sum \left(C_{\mathrm{A}_{n-1}}\right)_{uv}\left(C_{\mathrm{A}_{k-1}}^{-1}\right)_{st} m_u^{(s)} m_v^{(t)}}}{\prod_{i=1}^{n-1}\prod_{s=1}^{k-1}(q)_{m_i^{(s)}}} q^{\sum_{s=k_0+1}^{k-1}(s-k_0)m_j^{(s)} - \frac{k_j}{k}\sum_{s=1}^{k-1} s m_j^{(s)}},$$

where the $\sum_\bullet$ is taken over $\left(m_i^{(s)}\right) \in M_{(k-1)\times(n-1)}(\mathbb{Z}_{\geq 0})$ satisfying

$$(4.10.2) \qquad \overline{\Lambda} - \lambda = \sum_{i=1}^{n-1}\left(\sum_{s=1}^{k-1} s m_i^{(s)}\right)\alpha_i \pmod{k\mathring{Q}}.$$

Example 4.10.2. It is also interesting to note that Georgiev's formula (4.10.1) specialised to $n = 2$ was first obtained by Lepowsky and Primc in [LP85].

Writing $\lambda = \sum_{i=1}^{n} \lambda_i \overline{\Lambda}_i$ and observing

$$\sum_{v=1}^{n} \left(C^{-1}_{\mathrm{A}_{n-1}}\right)_{uv} = \frac{u(n-u)}{2},$$

we compute

$$\overline{\Lambda} - \lambda = \sum_{i=1}^{n} (\delta_{ij} k_j - \lambda_i) \overline{\Lambda}_i = \sum_{i=1}^{n} (\delta_{ij} k_j - \lambda_i) \frac{i(n-i)}{2} \alpha_i.$$

We have $c^{\Lambda}_{\lambda} = 0$ unless $\overline{\Lambda} - \lambda \in \mathring{Q}$, so it is clear $c^{\Lambda}_{\lambda} \neq 0$ when $\delta_{ij} k_j - \lambda_i \equiv 0 \pmod 2$. The condition (4.10.2) can be written as the system of congruences

$$\sum_{s=1}^{k-1} s m^{(s)}_i \equiv (\delta_{ij} k_j - \lambda_i) \frac{i(n-i)}{2} \pmod k \quad \text{for} \quad i = 1, \dots, n-1. \tag{4.10.3}$$

We are now ready to prove Proposition 4.0.1.

Proof Fix any $j = 1, \dots, k-1$ and suppose $\Lambda = k_0 \Lambda_0 + k_j \Lambda_j$ for some $k_0, k_j \in \mathbb{Z}_{\geq 0}$ such that $k_0 + k_j = k \geq 2$. The hypothesis guarantees $i(n-i)$ is invertible in $\mathbb{Z}/k\mathbb{Z}$ for $i = 1, \dots, n-1$. Therefore $i(n-i)(\delta_{ij} k_j - \lambda_i)/2$ runs through every congruence class modulo k exactly once as λ runs through $\mathring{P}$ subject to the conditions $\delta_{ij} k_j - \lambda_i \equiv 0 \pmod 2$ and $-k < \delta_{ij} k_j - \lambda_i < k$. Summing the left hand side of (4.10.1) over such $\lambda \in \mathring{P}$ we obtain

$$\left(\eta(\tau)\right)^n \sum_{\lambda} c^{\Lambda}_{\lambda} = q^{T_\Lambda - n/24} \sum_{(m^{(s)}_i) \in M_{(k-1)\times(n-1)}(\mathbb{Z}_{\geq 0})} \frac{q^{\frac{1}{2} \sum C_{uv} C^{-1}_{st} m^{(s)}_u m^{(t)}_v}}{\prod_{i=1}^{n-1} \prod_{s=1}^{k-1} (q)_{m^{(s)}_i}} q^{\sum_{s=k_0+1}^{k-1} (s-k_0) m^{(s)}_j - \frac{k_j}{k} \sum_{s=1}^{k-1} s m^{(s)}_j}, \tag{4.10.4}$$

where $T_\Lambda := T(n, k, k_0, j)$. The sum of string functions on the left hand side is a modular form of weight $(-1/2)^n$ with respect to $\Gamma(s)$ where $s = \operatorname{lcm}\{k, n, n+k, n-1\}$ by Theorem 4.6.3. Recalling Example 2.1.9, $\left(\eta(\tau)\right)^n$ is a modular form of weight $(1/2)^n$ with respect to $\mathrm{PSL}_2(\mathbb{Z})$ we see the right hand side of (4.10.4) is a modular function with respect to $\Gamma(s)$.

Example 4.10.3. When $n = 2$, we are considering the string functions of the unique irreducible $A^{(1)}_1$-module of highest weight $\Lambda = k_0 \Lambda_0 + k_1 \Lambda_1$. The greatest common divisor condition is satisfied for all $k \geq 2$. The sum in (4.10.4) is simply over all $\lambda \in \mathring{P}$ with $k_1 - \lambda_1 \equiv 0 \pmod 2$ and $-k < k_1 - \lambda_1 < k$. It simplifies to

$$\left(\eta(\tau)\right)^2 \sum_{\lambda} c^{\Lambda}_{\lambda} = q^{T_\Lambda - 1/12} \sum_{n \in (\mathbb{Z}_{\geq 0})^{k-1}} \frac{q^{\frac{1}{2} n^t \left(C_{\mathrm{A}_1} \otimes C^{-1}_{\mathrm{A}_{k-1}}\right) n - n C^{-1}_{\mathrm{A}_{k-1}} e_{k_0}}}{(q)_{n_1} \dots (q)_{n_{k-1}}}.$$

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Appendix A

A.1 Cartan matrices of ADE and T type

$$C_{A_n} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & \ddots & 0 \\ 0 & -1 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$C_{D_n} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & \ddots & 0 & 0 \\ 0 & \ddots & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$

$$C_{T_n} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & \ddots & 0 \\ 0 & -1 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

$$C_{E_6} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

$$C_{\mathrm{E}_7} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$$C_{\mathrm{E}_8} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

A.2 Indexed Dynkin diagrams of ADE and T type

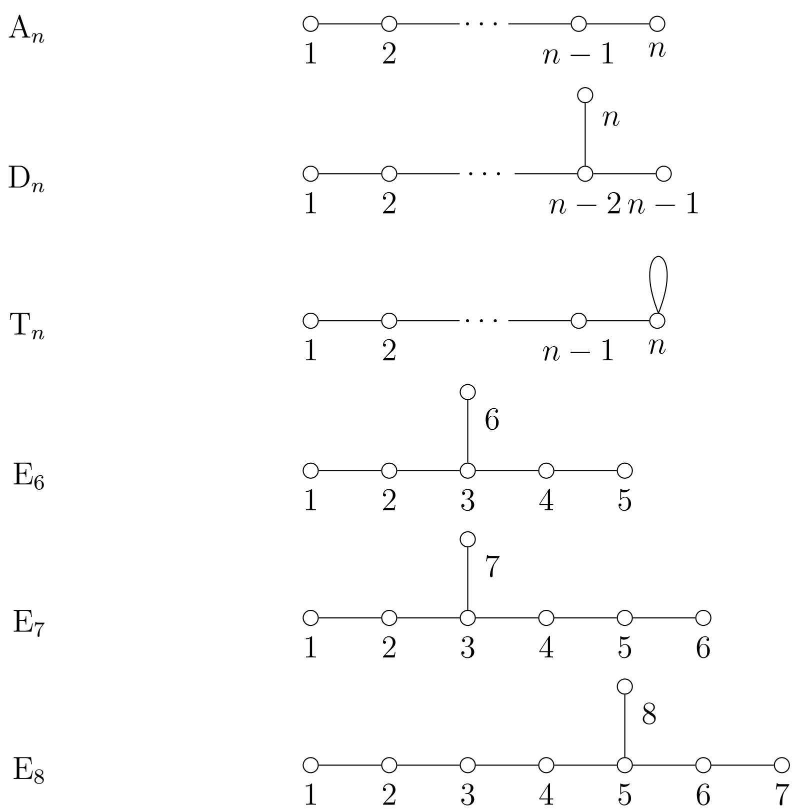

Figure A.1: Dynkin diagrams with indexed vertices.

A.3 Non-twisted extended Dynkin diagrams of ADE type

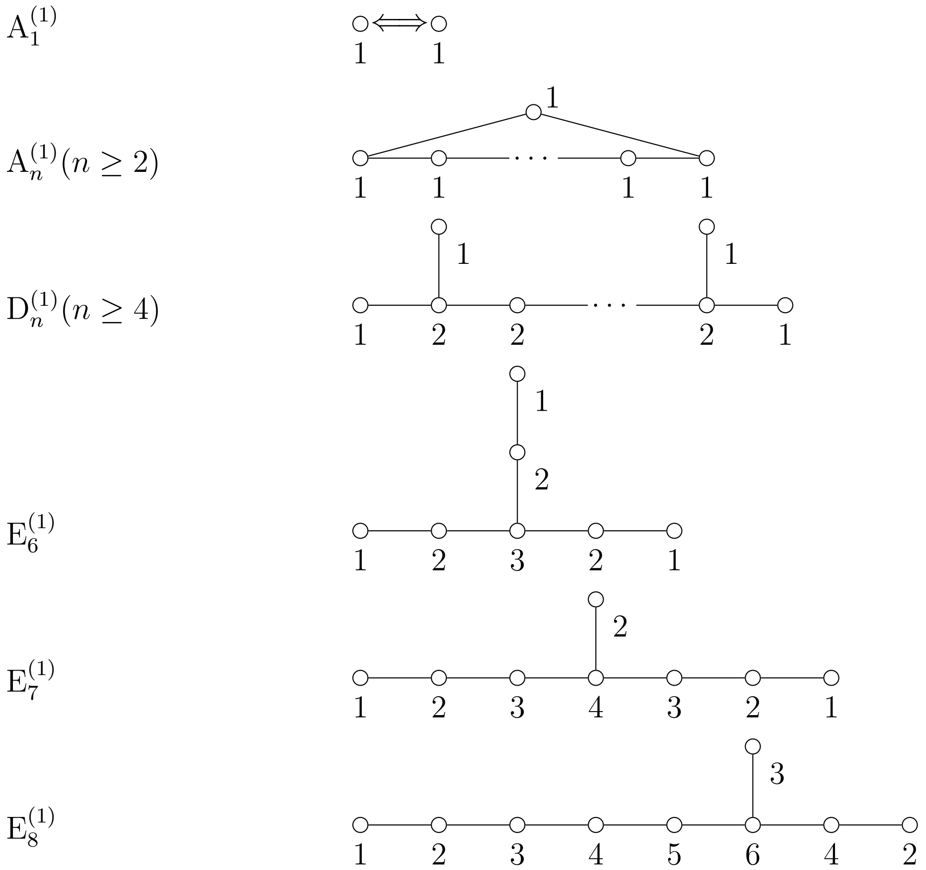

Figure A.2: Extended Dynkin diagrams with Dynkin labels.