Partition asymptotics and the polylogarithm

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Partitions

A **partition** of a positive integer n is a non-increasing sequence of positive integers whose sum is equal to n.

Each integer appearing in a partition of n is called a **part**.

The partitions of 4 are

$$\begin{aligned} 1+1+1+1 \\ 2+1+1 \\ 2+2 \\ 3+1 \\ 4. \end{aligned}$$

Partition function

Let $p(n)$ denote the number of partitions of n.

$$\begin{aligned} p(1) &= 1 \\ p(10) &= 42 \\ p(100) &= 190569292 \\ p(1000) &= 24061467864032622473692149727991 \\ &\vdots \end{aligned}$$

Hardy and Ramanujan asked if there was an asymptotic formula for $p(n)$?

Major MacMahon

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Rademacher later in 1937 refined this to

$$p(n) = \frac{2\pi}{(24n-1)^{\frac{3}{4}}} \sum_{c=1}^{\infty} \frac{A_c(n)}{c} I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6c}\right),$$

where $I_{\frac{3}{2}}$ denotes the I-Bessel function and $A_c(n)$ is a particular Kloosterman sum.

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$$\{1^3, 2^3, 3^3, 4^3, 5^3 \ldots\}.$$

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Around 1918, Hardy and Ramanujan conjectured formulae for partition asymptotics for **restricted** partitions.

Consider when each part of the partition is a dth power.

For example if $d = 3$, the set of possible parts is

$$\{1^3, 2^3, 3^3, 4^3, 5^3 \ldots\}.$$

Let $p_d(n)$ denote the number of such partitions.

How should asymptotics for $p_d(n)$ compare with those of $p(n)$?

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Natural question: are there simpler asymptotics for $p_d(n)$ that use functions that naturally appear in analytic number theory?

Say Γ-functions and Riemann ζ-functions?

Vaughan and Gafni's results (2015–2016)

Theorem 1 (Gafni, 2016 and Vaughan, 2015 when $d = 2$)

Fix $d \geq 2$. Let n be a sufficiently large integer and X and Y satisfy

$$X \sim \left(\frac{1}{d^2}\zeta\Big(1+\frac{1}{d}\Big)\Gamma\Big(\frac{1}{d}\Big)\right)^{-\frac{d}{d+1}} n^{\frac{d}{d+1}}$$

and

$$Y \sim \frac{d+1}{2d^3}\zeta\Big(1+\frac{1}{d}\Big)\Gamma\Big(\frac{1}{d}\Big)^{\frac{d}{d+1}} n^{\frac{1}{d+1}}.$$

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Then for each $J \in \mathbb{N}$, there exist $c_1, \ldots, c_J \in \mathbb{N}$ [independent of n] such that

$$p_d(n) = \frac{\exp\left(\frac{d+1}{d^2}\zeta\left(1+\frac{1}{d}\right)\Gamma\left(\frac{1}{d}\right)X^{\frac{1}{d}} - \frac{1}{2}\right)}{(2\pi)^{\frac{d+2}{2}} X^{\frac{3}{2}} Y^{\frac{1}{2}}} \times \left(\pi^{\frac{1}{2}} + \sum_{j=1}^{J-1} c_j Y^{-j} + O\left(Y^{-J-1}\right)\right).$$

Digesting Gafni's asymptotic

Crudely, Gafni's result says

$$p_d(n) \sim \mathcal{C}_1 n^{-\frac{3d+1}{2(d+1)}} \exp\left(\mathcal{C}_2 n^{\frac{1}{d+1}}\right).$$

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Compare with the shape with H–R asymptotic $[d = 1]$,

$$p(n) \sim \frac{1}{4\sqrt{3}n} \exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

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Pose the challenge of obtaining asymptotics for the partition function p_f, where $f \in \mathbb{Z}[y]$.

Observe that $p_f(n)$ counts the number of partitions of n such that each part belongs to

$$\{f(n) : n \in \mathbb{N}\}.$$

Set-up

We establish asymptotic formula for p_f where $f \in \mathcal{A} \subset \mathbb{Z}[y]$.

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Let

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such that $(a_d, \ldots, a_0) = 1$.

Factor,

$$f(y) - a_0 = y \prod_{j=0}^{d-1} (y + \alpha_j)$$

for some $\alpha_j \in \mathbb{C}$. Let $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_{d-1}, 0)$.

Consider the ζ–function

$$\zeta_d\big((s_1,\ldots,s_d);(\alpha_1,\ldots,\alpha_{d-1},0)\big) := \sum_{n=1}^{\infty} \frac{1}{(n+\alpha_1)^{s_1}\cdots(n+\alpha_{d-1})^{s_{d-1}}\,n^{s_d}},$$

where $s_j \in \mathbb{C}$ and $\alpha_j \in \mathbb{C}\setminus\mathbb{R}_{\leq -1}$.

Matsumoto–Weng Zeta function

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where $s_j \in \mathbb{C}$ and $\alpha_j \in \mathbb{C}\setminus\mathbb{R}_{\leq -1}$.

Here

$$(n+\alpha_j)^{s_j} = \exp\big(-s_j\log(n+\alpha_j)\big)$$

with

$$-\pi < \arg(n+\alpha_j) \leq \pi.$$

This series is clearly well defined and absolutely convergent in the region

$$\Re(s_1+\cdots+s_d) > 1.$$

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For $\lambda, s \in \mathbb{C}$ with $\Re s > 0, \lambda \neq 0, |\arg \lambda| < \pi$ and $-\Re s < C < 0$ we have the Mellin–Barnes integral formula

$$\Gamma(s)(1+\lambda)^{-s} = \frac{1}{2\pi i} \int_{(C)} \Gamma(s+z)\Gamma(-z)\lambda^z dz.$$

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Matsumoto and Weng used this formula to meromorphically continue $\zeta(s, \boldsymbol{\alpha})$ to all of $\mathbb{C}$.

Properties of Matsumoto–Weng Zeta function cont.

No Functional equation!

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Has an evaluation

$$\zeta(0, \boldsymbol{\alpha}) = \zeta(0) - \frac{1}{d} \sum_{j=1}^{d-1} \alpha_j.$$

This will appear in the main terms of our asymptotic formula.

Partitions into a values of a fixed polynomial

Given a large $n \in \mathbb{N}$, let

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where the constants depend on f.

Further let

$$\mathcal{C} \sim \frac{1}{d}\zeta\left(\frac{1+d}{d}\right)\Gamma\left(\frac{1}{d}\right)\left(\frac{X}{a_d}\right)^{1/d}.$$

Our Theorem

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$$a_0 \geq 0, \quad a_1 = 0, \quad \frac{a_{d-1}}{a_d} := \sum_{j=1}^{d-1} \alpha_j < \frac{d}{2}, \quad \frac{a_0}{a_d} < 1$$

and f is non-constant as a function modulo p for all primes $p \leq d$. Let $n, X, Y \in \mathbb{R}$ be as above and $0 < R < 1$ fixed.

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and f is non-constant as a function modulo p for all primes $p \leq d$. Let $n, X, Y \in \mathbb{R}$ be as above and $0 < R < 1$ fixed. Then, for any $1 < J < dR$, there exist $w_1, \cdots, w_{J-1} \in \mathbb{R}$ (independent of n) such that

$$p_{\mathcal{A}_f}(n) = \frac{1}{2\pi a_d^{\zeta(0,\boldsymbol{\alpha})}} \frac{\exp\left(\mathcal{C} + \frac{n}{X}\right)}{X^{1-\zeta(0,\boldsymbol{\alpha})} Y^{\frac{1}{2}}} \left(\sqrt{\pi} + \sum_{q=1}^{J-1} w_q Y^{-q} + O_{f,R}(Y^{-J})\right)$$

as $n \to \infty$.

Comparing the shape

Gafni/Vaughan:

$$p_d(n) \sim \mathcal{C}_1 n^{-\frac{3d+1}{2(d+1)}} \exp\left(C_2 n^{\frac{1}{d+1}}\right).$$

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Our asymptotic:

$$p_{\mathcal{A}_f}(n) \sim \mathcal{C}_3 n^{-\frac{2d(1-\zeta(0,\boldsymbol{\alpha}))+1}{2(d+1)}} \exp\left(\mathcal{C}_4 n^{\frac{1}{d+1}}\right).$$

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Using the evaluations for $\zeta(0,\boldsymbol{\alpha})$ and $\zeta(0) = -1/2$ we have:

$$p_{\mathcal{A}_f}(n) \sim \mathcal{C}_3 n^{-\frac{3d+1+(\alpha_1+\cdots+\alpha_{d-1})}{2(d+1)}} \exp\left(\mathcal{C}_4 n^{\frac{1}{d+1}}\right).$$

Taking $\alpha_1 = \cdots := \alpha_{d-1} = 0$ recovers the main terms of Gafni/Vaughan asymptotic.

Goals for the rest of talk

Explain the strategy behind the proofs of Vaughan and Gafni.

- Circle method

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Explain the new ideas behind the proof of our theorem

- Matsumoto–Weng Zeta function
- Polylogarithm identity
- More general estimates for Waring-type major arcs
- Roots of the polynomial control the main terms

Generating functions for dth powers

Generating function for $p_d(n)$ is given by

$$\Psi_d(z) := \sum_{n=0}^{\infty} p_d(n) z^n = \prod_{n=1}^{\infty} (1 - z^{n^d})^{-1}.$$

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The logarithm is given by

$$\Phi_d(z) := \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{j} z^{jn^d}.$$

Clearly

$$\Psi_d(z) = \exp\big(\Phi_d(z)\big).$$

Useful quantities

Recall that $X \sim \mathcal{C} n^{\frac{d}{d+1}}$, for some constant $\mathcal{C}$ depending on d.

We define the useful quantity $\rho := \exp\left(-1/X\right)$.

Take $z := \rho e(\Theta)$ where $\Theta \in \mathbb{R}$. Here $e(\Theta) := \exp(2\pi i \Theta)$.

Hardy–Littlewood circle method strategy for dth powers

Apply Cauchy's Theorem to recover the coefficients of the generating function Ψ_d:

$$p_d(n) = \int_0^1 \rho^{-n} \exp\left(\Phi_d\left(\rho e(\Theta)\right) - 2\pi i n\Theta\right) d\Theta.$$

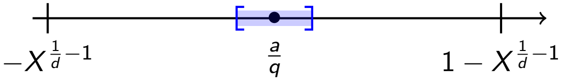
The integrand has period 1 with respect to Θ.

Instead integrate over $\mathcal{U} := (-X^{\frac{1}{d}-1}, 1 - X^{\frac{1}{d}-1}]$.

From which parts of $\mathcal{U}$ do the main contributions come from?

H–L circle method strategy for dth powers continued

Basic idea of the H–L circle method is that main contributions come from small intervals around rational numbers a/q whose denominator is bounded.



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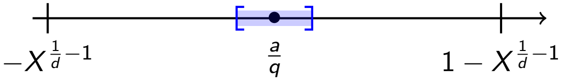

For $a, q \in \mathbb{N}$ with $(a, q) = 1$, define

$$\mathfrak{M}(q, a) := \left\{ \Theta \in \mathcal{U} : \left| \Theta - \frac{a}{q} \right| \le q^{-1} X^{\frac{1}{d}-1} \right\},$$

and let

$$\mathfrak{M} := \bigcup_{\substack{1 \le a \le q \le X^{\frac{1}{d}} \\ (a,q)=1}} \mathfrak{M}(q, a).$$

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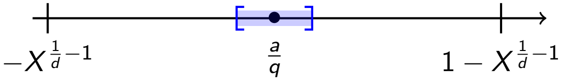

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These are the **major arcs**.

H–L circle method for dth powers continued

The minor arcs are

$$\mathfrak{m} := \mathcal{U} \setminus \mathfrak{M}.$$

Minor arcs can be easily handled using an argument with Dirichlet's Approximation Theorem.

H–L circle method for dth powers continued

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Minor arcs can be easily handled using an argument with Dirichlet's Approximation Theorem.

Atypical circle method application. All the main terms come from $\mathfrak{M}(1, 0)$.

Other major arcs can be subsumed into the error coming from the minor arcs.

Extracting main terms and errors for dth powers

Anatomy of the proof:

$$\Bigg(\underbrace{\int_{\mathfrak{M}(1,0)}}_{\text{main terms}} + \underbrace{\int_{\mathfrak{M}\setminus\mathfrak{M}(1,0)} + \int_{\mathfrak{m}}}_{\text{error}} \Bigg) \rho^{-n} \exp\Big(\Phi_d\big(\rho e(\Theta)\big) - 2\pi i n\Theta\Big) d\Theta.$$

Let us focus on the extraction of the main terms of the asymptotic and controlling the resulting error.

Sufficient to study to study the first integral over the smaller interval $\big[-\frac{3}{8\pi X}, \frac{3}{8\pi X}\big]$.

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Write

$$\Phi_d\big(\rho e(\Theta)\big) = \sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{j}\exp\left(-jn^k\Big(\frac{1}{X}-2\pi i\Theta\Big)\right).$$

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Using a Mellin transform we obtain

$$\Phi_d\big(\rho e(\Theta)\big) = \frac{1}{2\pi i}\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{j}\int_{c-i\infty}^{c+i\infty}\Gamma(s)j^{-s}n^{-ds}\Big(\frac{X}{1-2\pi i\Theta X}\Big)^{s}ds,$$

for $c > 0$.

Main terms continued

For $c > 1/d$ we can swap the sums and integration to obtain

$$\Phi_d\big(\rho e(\Theta)\big) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \zeta(1+s)\zeta(ds)\Big(\frac{X}{1-2\pi i\Theta X}\Big)^s \Gamma(s)ds.$$

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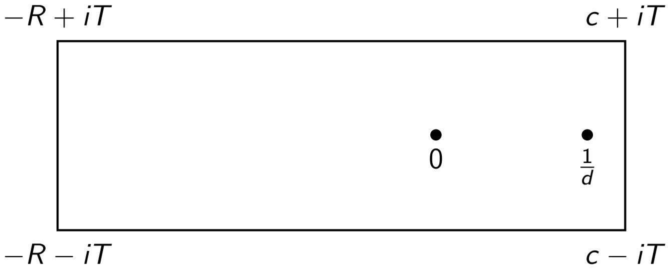
Let $R \geq 3/2$ be a parameter and extend the region of integration to a rectangle



Main terms continued

Horizontal sides:

$$\int_{c+iT}^{-R+iT} \longrightarrow 0 \quad \text{and} \quad \int_{-R-iT}^{c+iT} \longrightarrow 0 \quad \text{as} \quad T \to \infty.$$

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Left vertical side can be controlled by a double application of the ζ-functional equation and Stirling's formula to

$$\zeta(s+1)\zeta(ds)\Gamma(s).$$

One can choose an optimal R to minimise the error.

Main terms continued

Double pole of $\Gamma(s)\zeta(s+1)$ at $s=0$

Simple pole of $\zeta(ds)$ at $s=\dfrac{1}{d}$

Residue analysis

$$\Phi_d\big(\rho e(\Theta)\big) \sim \frac{1}{d}\zeta\Big(1+\frac{1}{d}\Big)\Gamma\Big(\frac{1}{d}\Big)\Big(\frac{X}{1-2\pi i\Theta X}\Big)^{\frac{1}{d}} - \frac{1}{2}\log\Big(\frac{(2\pi)^d X}{1-2\pi i\Theta X}\Big) + \cdots$$

This is one of the main inputs used in the H–L circle method computation of Gafni and Vaughan.

Generating functions for polynomials

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Clearly

$$\Psi_f(z) = \exp\left(\Phi_f(z)\right).$$

Again apply Cauchy's Theorem

$$p_f(n) = \int_0^1 \rho^{-n} \exp\Big(\Phi_f\left(\rho e(\Theta)\right) - 2\pi i n \Theta\Big) d\Theta.$$

Main terms and the polylogarithm

Let

$$x := \frac{1 - 2\pi i \Theta X}{X}.$$

Then Φ_f can be re-written as

$$\Phi_f\big(\rho e(\Theta)\big) = \sum_{j=1}^{\infty} \frac{\exp(-j a_0 x)}{j} \sum_{n=1}^{\infty} \exp\big(-j(f(n) - a_0)x\big).$$

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Let

$$x := \frac{1 - 2\pi i \Theta X}{X}.$$

Then Φ_f can be re-written as

$$\Phi_f\left(\rho e(\Theta)\right) = \sum_{j=1}^{\infty} \frac{\exp(-j a_0 x)}{j} \sum_{n=1}^{\infty} \exp\left(-j(f(n) - a_0)x\right).$$

Using the Cahen–Mellin transform

$$\Phi_f\left(\rho e(\Theta)\right) = \frac{1}{2\pi i} \sum_{j=1}^{\infty} \frac{\exp(-j a_0 x)}{j} \sum_{n=1}^{\infty} \int_{(c)} \Gamma(s) \big(j(f(n) - a_0)x\big)^{-s} ds.$$

Here we choose $c := 1/d + \varepsilon$ for any $\varepsilon > 0$.

Main terms and the polylogarithm cont.

Factoring the polynomial,

$$\Phi_f\left(\rho e(\Theta)\right) = \frac{1}{2\pi i}\sum_{j=1}^{\infty}\exp(-ja_0x)$$
$$\times\sum_{n=1}^{\infty}\int_{(c)} j^{-s-1}\Gamma(s)x^{-s}a_d^{-s}n^{-s}\prod_{l=1}^{d-1}(n+\alpha_l)^{-s}ds.$$

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Interchanging summation on n and the integral yields

$$\Phi_f\left(\rho e(\Theta)\right) = \frac{1}{2\pi i}\sum_{j=1}^{\infty}\int_{(c)}\frac{\exp\left(-ja_0x\right)}{j^{s+1}}\Gamma(s)x^{-s}a_d^{-s}\zeta(s,\boldsymbol{\alpha})ds.$$

Main terms and the polylogarithm cont.

Summing over j

$$\Phi_f\left(\rho e(\Theta)\right) = \frac{1}{2\pi i}\int_{(c)} \mathrm{Li}_{s+1}(e^{-a_0 x})\Gamma(s)x^{-s}a_d^{-s}\zeta(s,\alpha)ds, \qquad (1)$$

where $\mathrm{Li}_s(z)$ is the polylogarithm function

$$\mathrm{Li}_s(z) := \sum_{k=1}^{\infty}\frac{z^k}{k^s} \quad \text{for} \quad s\in\mathbb{C} \quad \text{and} \quad |z|<1.$$

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The main idea now is to try and write this as in integral involving Γ and ζ functions.

Polylogarithm identity

We have the polylogarithm identity:

$$\boxed{\mathrm{Li}_s(e^{\mu}) = \Gamma(1-s)(-\mu)^{s-1} + \sum_{k=0}^{\infty} \frac{\zeta(s-k)}{k!}\mu^k,}$$

valid for all $|\mu| < 2\pi$ and $s \neq 1, 2, 3, \ldots$.

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Reduced to analysing

$$\Phi_f\left(\rho e(\Theta)\right) = \frac{1}{2\pi i}(I_1 + I_2)$$

where

$$I_1 := \int_{(c)} \Gamma(s)\Gamma(-s)\left(\frac{a_0}{a_d}\right)^s \zeta(s,\alpha)ds$$

$$I_2 := \int_{(c)} \sum_{k=0}^{\infty} \frac{(-1)^k \zeta(s+1-k)}{k!}\Gamma(s)a_0^k a_d^{-s} x^{k-s}\zeta(s,\alpha)ds.$$

Now I_2 involves an infinite sum.

Main terms for polynomial case

A careful residue analysis of I_2 yields the main terms that are inputted into the H–L circle method.

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This is one of the main reasons why the error in our asymptotic is limited by the degree of the polynomial.

Major arc estimates

We still require major arc estimates in order to control the contributions from the other major arcs.

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This required us to generalise the major arc estimates used in Waring's problem.

Major arc estimates continued

Recall

$$\mathcal{F}(\Theta) := \sum_{y=1}^{U} e\big(\Theta f(y)\big).$$

Lemma 3 (D. and Robles, 2017)

Let $f(y) \in \mathbb{Z}[y]$ be as in our Theorem. Suppose $a, q \in \mathbb{N}$ such that $(a,q) = 1$ and $\Theta = a/q + \beta$ where $|\beta| \leq 1/q$.

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Let $f(y) \in \mathbb{Z}[y]$ be as in our Theorem. Suppose $a, q \in \mathbb{N}$ such that $(a, q) = 1$ and $\Theta = a/q + \beta$ where $|\beta| \leq 1/q$. Then for any $\varepsilon > 0$ we have

$$\mathcal{F}(\Theta) - \mathcal{V}(\Theta, q, a, f) \ll_{f,\varepsilon} q^{1-2^{1-d}+\varepsilon}(1 + U^d|\beta|)^{\frac{1}{2}},$$

where

$$\mathcal{S}(q, a, f) := \sum_{y=1}^{q} e\Big(\frac{af(y)}{q}\Big)$$

$$v_f(\beta) := \int_0^U e\big(\beta f(\gamma)\big)d\gamma$$

$$\mathcal{V}(\Theta, q, a, f) := q^{-1}\mathcal{S}(q, a, f)v_f(\Theta - a/q).$$

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